Universal Trees and Quasi-Polynomial Algorithms for Solving Parity Games

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Parity games have played a fundamental role in automata theory, logic, and their applications to verification and synthesis since early 1990's. Solving parity games is polynomial-time equivalent to checking *emptiness of automata on infinite trees* and to the *modal mu-calculus model checking*. It is a long-standing open question whether there is a polynomial-time algorithm for solving parity games. The quest for a polynomial-time algorithm has not only brought diverse algorithmic techniques to the theory and practice of verification and synthesis, but it has also significantly contributed to resolving long-standing open problems in other research areas, such as Markov Decision Processes and Linear Programming.

All algorithms for solving parity games that were known until 2016 required time that was exponential in the most important parameter of a parity game—the number of distinct *priorities*. The major breakthrough was achieved by Calude, Jain, Khoussainov, Li, and Stephan in 2017, who have given the first quasi-polynomial algorithm and established that parity games are in FPT (fixed-parameter tractable). Two other quasi-polynomial algorithms for solving parity games were subsequently devised by Jurdziński and Lazić, 2017, and by Lehtinen, 2018, and a space-efficient version of Calude et al.'s algorithm was given by Fearnley, Jain, Schewe, Stephan, and Wojtczak, 2017. The conceptual and technical toolkits used by all the three algorithms seem rather distinct: the breakthrough result of Calude et al. was based on computing *play summaries* by *succinct counting*, Jurdziński and Lazić have devised a *succinct coding* of *ordered trees* and applied it to the *progress measure lifting* algorithm, and Lehtinen has developed novel concepts of *register games* and the *register index*.

In this talk we first focus on presenting the technical insights of the quasi-polynomial algorithm for solving parity games that is based on progress measure lifting and succinct coding of ordered trees. Following Czerwiński, Daviaud, Fijalkow, Jurdziński, Lazić, and Parys, 2018, we then argue that *universal ordered trees*—implicit in the succinct tree-coding result of Jurdziński and Lazić—offer a unifying perspective on the three distinct quasi-polynomial algorithms. Moreover, the analysis of universal trees leads to an automata-theoretic quasi-polynomial lower bound that forms a barrier that all the existing approaches, as well as other possible techniques that follow the separation approach, must overcome in the quest for a polynomial-time algorithm for solving parity games.

More specifically, we argue that the techniques underlying all the three quasi-polynomial algorithms can be interpreted as constructions of automata on infinite words that are of quasi-polynomial size and that facilitate solving parity games by the *separation approach* formalized by Bojańczyk and Czerwiński, 2018, and implicit in

the work of Bernet, Janin, and Walukiewicz, 2002. In particular, we point out how such *separating automata* arise in a very natural way from universal ordered trees. Then we present two lower bounds: one is a quasi-polynomial lower bound on the size of universal trees that nearly matches (up to a small polynomial factor) the succinct tree-coding upper bound of Jurdziński and Lazić, and the other establishes that the set of states in every separating automaton contains leaves of some universal tree, which implies that every separating automaton is of at least quasi-polynomial size.

Keywords: Parity games · Quasi-polynomial algorithms · Progress measures · Universal ordered trees · Separating automata · Lower bounds