Primitivity of random matrix sets and the synchronizing probability function

Costanza Catalano

(joint work with Raphaël M. Jungers (UCLouvain))

Gran Sasso Scince Institute, L'Aquila, Italy

RP'18, Marseille, September 25th 2018

What is a **primitive** set of matrices?

Primitive matrix:

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad M^3 = \begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 2 \\ 3 & 2 & 1 & 2 \end{pmatrix} > 0$$

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Primitive set (of matrices):

$$\mathcal{M} = \Bigg\{ \underbrace{\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{M_1}, \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{M_2} \Bigg\}, \quad M_1 M_2 M_2 M_1 M_2 M_2 M_1 = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 \end{pmatrix} > 0$$

A primitive set is

a finite set of nonnegative matrices that admits an entrywise **positive product**.

- Introduced by Protasov, Voynov in 2012.

A little bit less than primitivity: **scrambling** sets

Scrambling set (of matrices):

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A **scrambling** set is

a finite set of nonnegative matrices that admits a product with an entrywise **positive column**.

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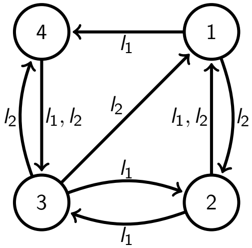
- $\mathcal{M} = \{M_1, \ldots M_m\}$, $\forall\, i\ M_i$ has **no zero-rows nor zero-columns** and $M = \sum_i M_i$ is **irreducible**, then

$$\text{Primitivity} \quad \Leftrightarrow \quad \text{Scrambing}$$

Example I: Labelled directed graphs

$\mathcal{D} = (V, E)$:

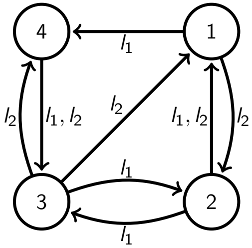
- $V = \{1, \ldots, n\}$, $n \in \mathbb{N}$;
- $E \subseteq V \times V$;
- labels $\mathcal{L} = \{l_1, \ldots, l_m\}$, $m \in \mathbb{N}$;
- every edge has **at least one label** (multiple labels allowed).



$V = \{1, 2, 3, 4\}$
$\mathcal{L} = \{l_1, l_2\}$

Example I: Labelled directed graphs

1. $\exists \mathbf{v} \in V$ and a **sequence of labels** $l = l_{i_1} \dots l_{i_k}$ s.t. **every node reaches v** following a **path labeled** by l? **Minimal length** of l?



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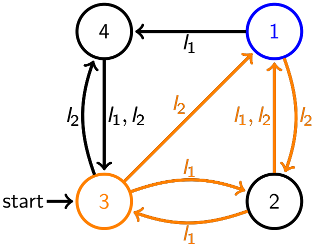

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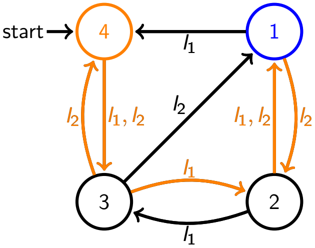

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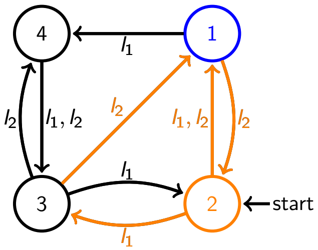

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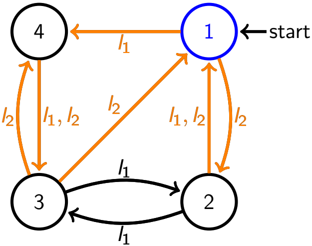

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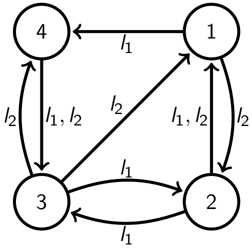

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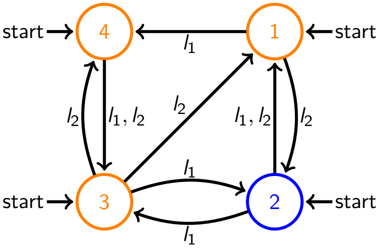

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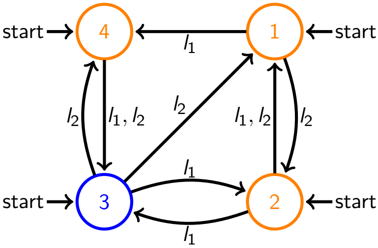

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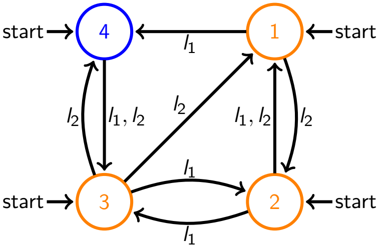

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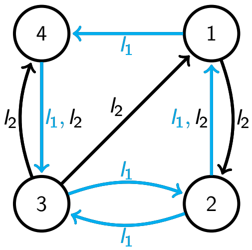

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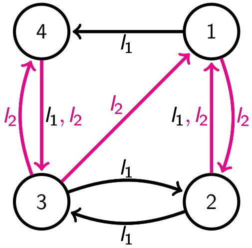

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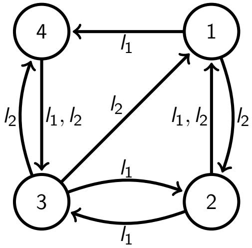

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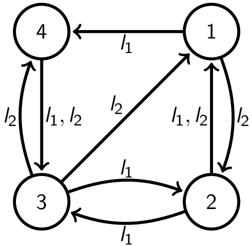

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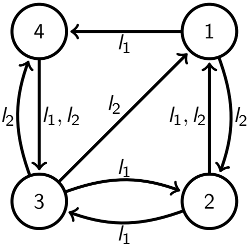
YES iff the set $\mathcal{M}$ is primitive. $|l|$ = length of the shortest positive product.



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Example II: Discrete time switching systems

$\mathcal{M} = \{M_1, \ldots, M_m\}$.

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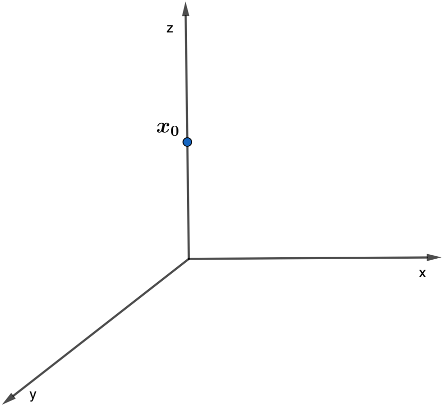

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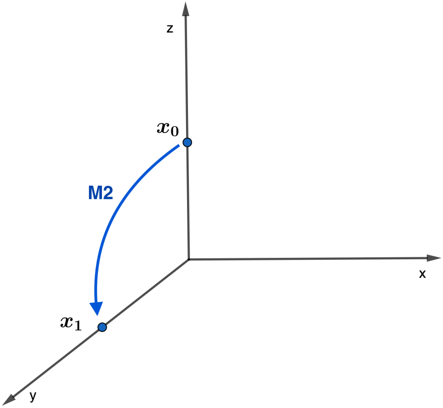

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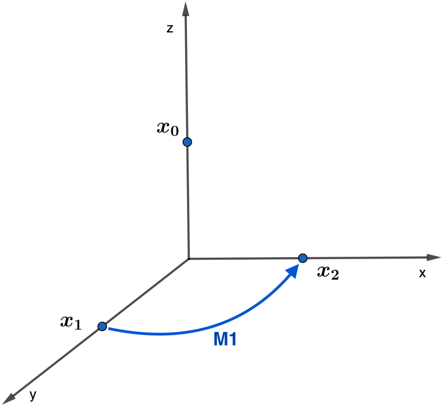

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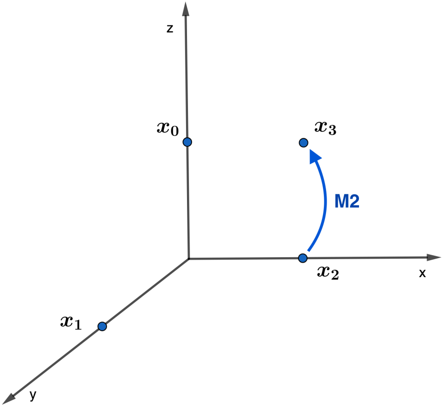

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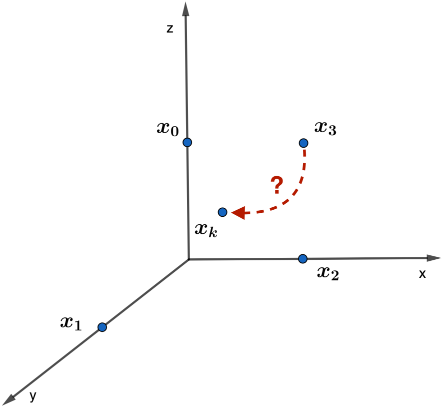

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Can I reach $\mathbb{R}_{>0}$?

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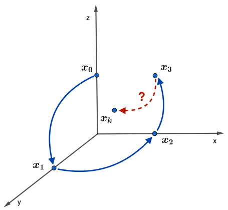

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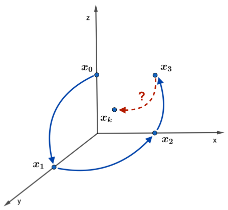

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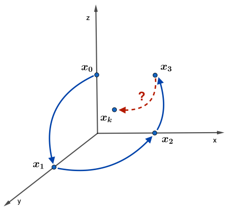

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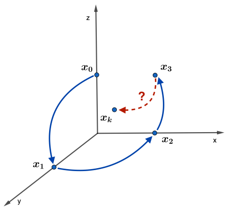
Example II: Discrete time switching systems



$$\begin{cases} x_{k+1} = M_{i_k} x_k & M_{i_k} \in \mathcal{M} \\ x_0 \in \mathbb{R}^n_{\geq 0} \end{cases}$$

$$\mathcal{M} = \Bigg\{ \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{M_1}, \underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{M_2} \Bigg\}$$

1. $\exists \mathbf{x_0} \in \{x, y, z\}$ and **sequence of matrices** of the system s.t. $\exists k : x_k$ **reaches** $\mathbb{R}^n_{>\mathbf{0}}$? What is the **minimal** k?

YES iff the set $\mathcal{M}$ is scrambling.
k = length of the shortest product with a positive column.

2. Can we choose a **sequence of matrices** of the system s.t. $\exists k : x_k$ **reaches** $\mathbb{R}^n_{>\mathbf{0}}$ for **any** $x_0 \in \mathbb{R}^n_{\geq 0}$? What is the **minimal** k?

YES iff the set is primitive.
k = length of the shortest positive product.

Complexity and bounds on the shortest positive product

Exponent of a primitive set

exp$(\mathcal{M})$= **length** of the **shortest positive** product of the set $\mathcal{M}$.

- Computing the exponent of a set is **NP-hard**. [Gerencsér, Gusev, Jungers, 2016]
- **Asymptotics** on the growth-rate of $exp(\mathcal{M})$ w.r.t. the matrix size n:
  - $\max_{\mathcal{M}}\big(exp(\mathcal{M})\big) \sim \sqrt[3]{3}e^{n}$ [Gerencsér, Gusev, Jungers, 2016]
  - $exp(\mathcal{M}) \leq (n^3 + 2n - 3)/3$ when every matrix of $\mathcal{M}$ has neither zero-rows nor zero-columns [Blondel, Jungers, Olshevsky, 2015]

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    Determining primitivity is decidable but **NP-hard**

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    Determining primitivity is decidable in **polynomial** time

Outline

What is the probability to generate a primitive set? What is their typical exponent?

- Random matrix sets:
  - Primitivity and shortest positive product
  - Scrambling and shortest column-positive product

Can we approximate the exponent of a set?

- A probabilistic tool for studying primitivity
- A new bound on the exponent of a class of primitive sets

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Boolean matrix products

The properties of being primitive or scrambling are **not** influenced by the **magnitude** of the **positive** entries of the matrices.

Assumption

Every positive entry in a matrix is set to 1.

$$M = \begin{pmatrix} 3 & 0 & 1 & 2 \\ 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 1 \\ 2 & 0 & 2 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \; M_1 M_2 M_2 M_1 M_2 M_2 M_1 = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Random matrix sets

Fix $m = \#$ of matrices of size $n \times n$

$$\begin{pmatrix} 1 & & \\ & & \end{pmatrix}, \qquad \begin{cases} = 1 & \text{with probability } p = p(n) \\ = 0 & \text{with probability } 1 - p \end{cases}$$

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$$m = 3, n = 4$$

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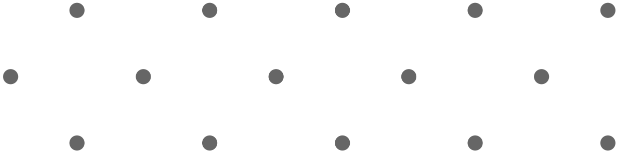
Equivalently: $m = \#$ of labels, $n = \#$ of vertices



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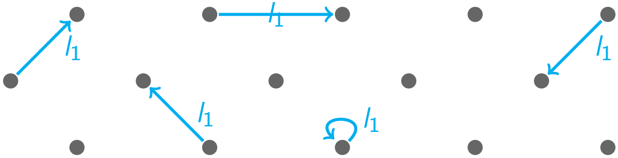

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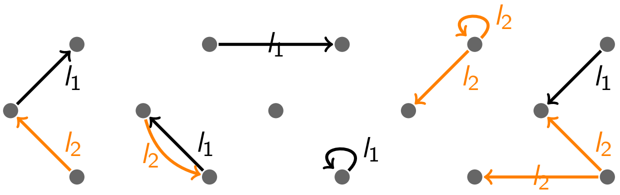

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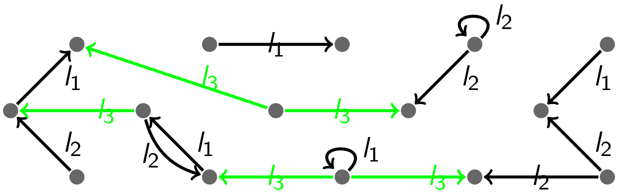

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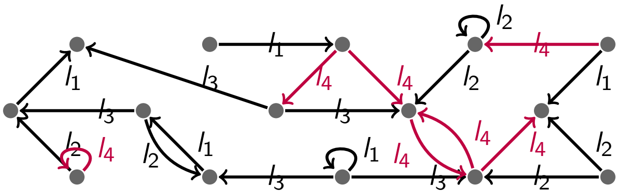

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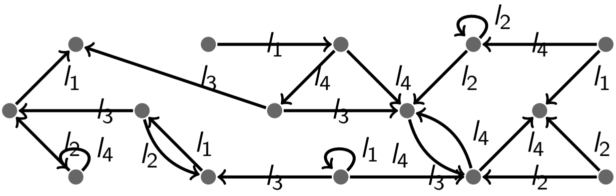

$m = 4, n = 15$

Primitivity of random sets: our result

$\mathcal{B}_{\mathbf{m}}^{\mathbf{n}}(\mathbf{p})$ = set of m matrices of size $n \times n$ sampled as seen before.

$$\begin{cases} = 1 & \text{with prob. } p = p(n) \\ = 0 & \text{with prob. } 1 - p \end{cases}$$

$\mathbf{exp}(\mathcal{B}_m^n(p))=$ length of its shortest positive product.

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Then as $n \to \infty$:

$$\mathbb{P}\big(\mathcal{B}^n_m(p) \text{ is primitive}\big) \longrightarrow \begin{cases} 1 & \text{if } np - \log n \to +\infty \\ (*) & \text{if } np - \log n \to c \in \mathbb{R} \\ 0 & \text{if } np - \log n \to -\infty \end{cases}$$

$$(*) \in \left[1-\left(1-e^{-2e^{-c}}\right)^m - me^{-2e^{-c}}\left(1-e^{-2e^{-c}}\right)^{m-1}, 1-\left(1-e^{-e^{-c}}\right)^m\right] \subset (0,1)$$

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Furthermore:

- If $np - \log n \to +\infty$: $exp(\mathcal{B}_m^n(p)) = O(n \log n)$ with **high probability**
- If $np - \log n \to c \in \mathbb{R}$: $exp(\mathcal{B}_m^n(p)) = O(n \log n)$ with **high probability** conditioned to the fact that every matrix of $\mathcal{B}_m^n(p)$ has neither zero-rows nor zero-columns

Scrambling property of random sets: our result

$\mathcal{B}^{\mathbf{n}}_{\mathbf{m}}(\mathbf{p})$ = set of m matrices of size $n \times n$ sampled as seen before.

$$\begin{cases} = 1 & \text{with prob. } p = p(n) \\ = 0 & \text{with prob. } 1 - p \end{cases}$$

scr$(\mathcal{B}^n_m(p))$= length of its shortest product with a positive column.

Then as $n \to \infty$:

$$\mathbb{P}\big(\mathcal{B}^n_m(p) \text{ is scrambling}\big) \longrightarrow \begin{cases} 1 & \text{if } np - \log n \to +\infty \\ (*) & \text{if } np - \log n \to c \in \mathbb{R} \\ 0 & \text{if } np - \log n \to -\infty \end{cases}$$

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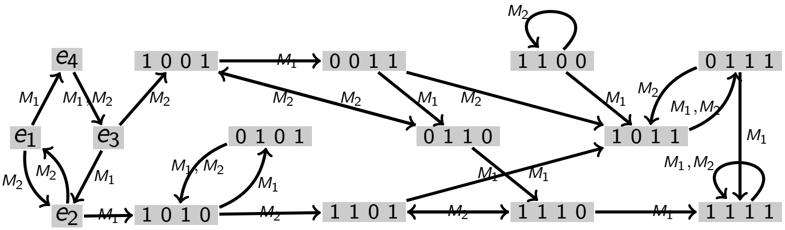
Primitivity as a 2-player game

$\mathcal{M} = \{M_1, \ldots, M_m\}$ a set of $n \times n$ matrices with neither zero-rows nor zero-columns (**NZ** matrices).

The directed graph $\mathcal{D} = (\mathcal{V}, \mathcal{E})$:

- $\mathcal{V} = \left\{ v \in \{0,1\}^n : v \neq (0,\ldots,0) \right\}$
- $v \to w$ labeled by M_k if $\mathbf{v}\,\mathbf{M_k} = \mathbf{w}$.

$$\left\{ M_1 = \begin{pmatrix} 0&0&0&1\\ 1&0&1&0\\ 0&1&0&0\\ 0&0&1&0 \end{pmatrix}, M_2 = \begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 1&0&0&1\\ 0&0&1&0 \end{pmatrix} \right\}, \quad \{e_1, e_2, e_3, e_4\} = \text{canonical basis}$$



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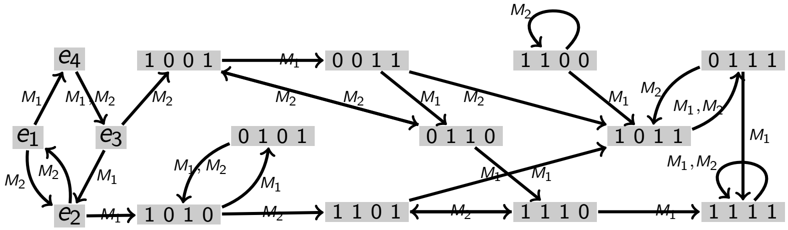
$\mathcal{M} = \{M_1, \ldots, M_m\}$ set of NZ-matrices. **Fix t ≥ 1 integer**.

The game's rules

1. Player B secretly chooses an initial vertex $e_i \in \{e_1, \ldots, e_n\}$.
2. Player A chooses a sequence of at most t matrices in $\mathcal{M}$.
3. Let $w = e_i^T M_{i_1} \cdots M_{i_t}$. A component of w is chosen uniformly at random: if it is $= 1$ Player A wins, otherwise Player B wins.



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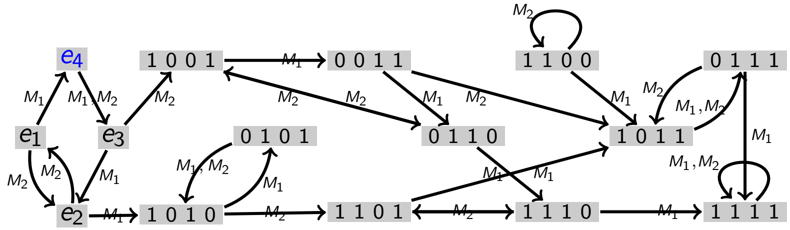
Player B: e_4



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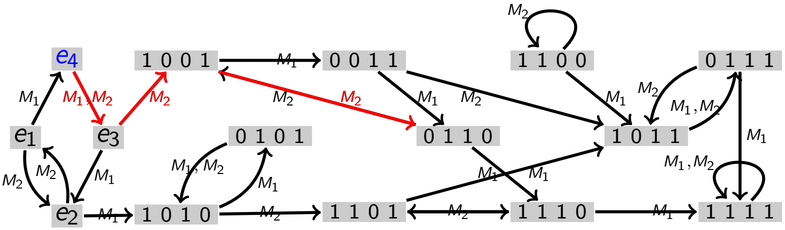

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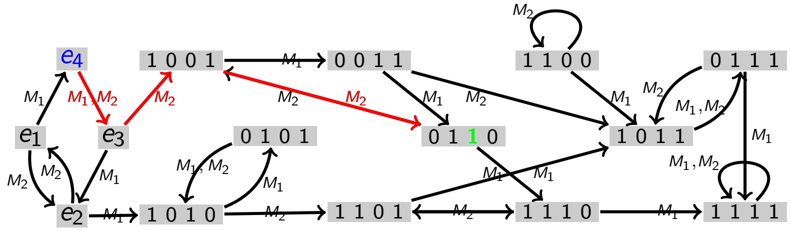

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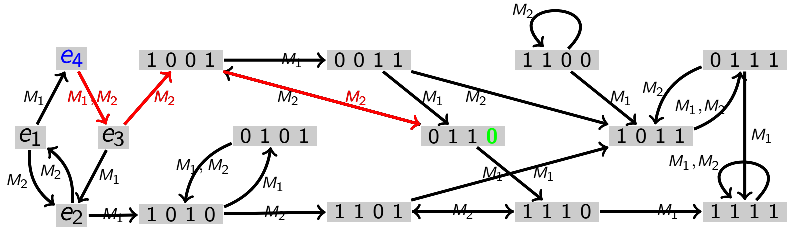

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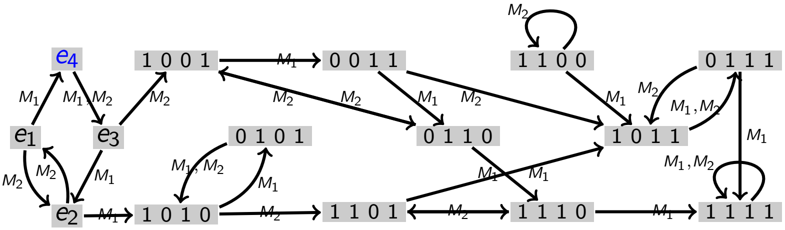

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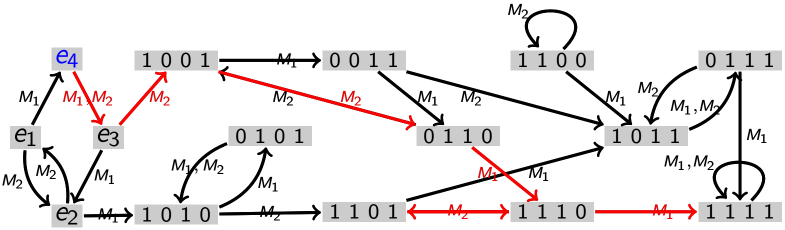
Player B: e_4 Player A: $M_1 M_2 M_2 M_1 M_2 M_2 M_1 > 0$ $(t = 7)$



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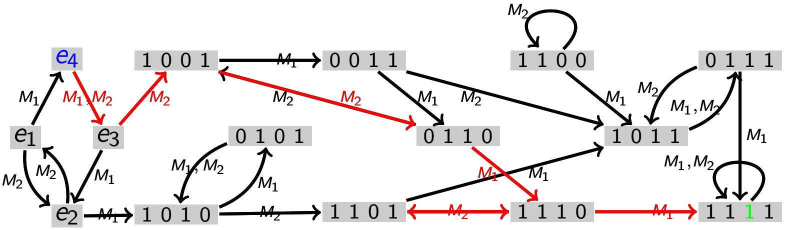

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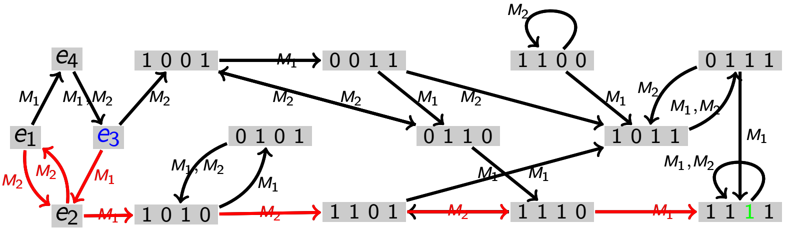
Player B: e_3 Player A: $M_1 M_2 M_2 M_1 M_2 M_2 M_1 > 0$ $(t = 7)$ Player A wins!
Independently of what Player B plays!



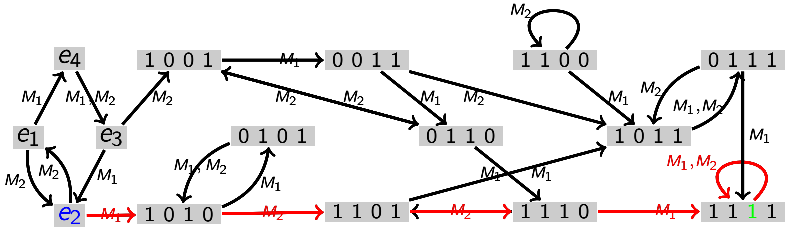
Primitivity as a 2-player game

$\mathcal{M} = \{M_1, \ldots, M_m\}$ set of NZ-matrices. Fix $t \geq 1$ integer.

The game's rules

1. Player B secretly chooses an initial vertex $e_i \in \{e_1, \ldots, e_n\}$.
2. Player A chooses a sequence of at most t matrices in $\mathcal{M}$.
3. Let $w = e_i^T M_{i_1} \cdots M_{i_t}$. A component of w is chosen uniformly at random: if it is $= 1$ Player A wins, otherwise Player B wins.

Player B: e_2 Player A: $M_1 M_2 M_2 M_1 M_2 M_2 M_1 > 0$ $(t = 7)$ Player A wins!
Independently of what Player B plays!



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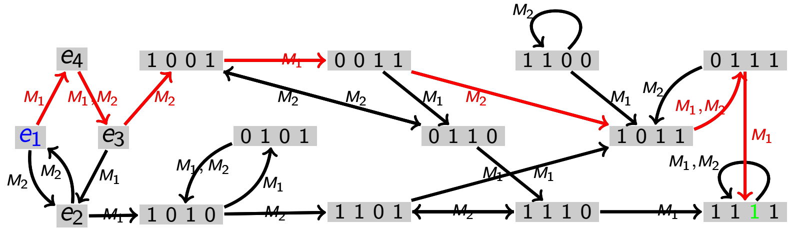

The Synchronizing Probability Function for primitive sets

The probability that Player A wins if they both play optimally is:

$$K_{\mathcal{M}}(t) = \min_{\substack{p \in \mathbb{R}^+ \\ p^T e = 1}} \left\{ \max_{M_{i_1} \cdot \ldots \cdot M_{i_l} \in \mathcal{M}^{\leq t}} \; p^T M_{i_1} \cdots M_{i_l} \left(\frac{\mathbf{e}}{n} \right) \right\}$$

where p is the initial distribution of player B, $\mathcal{M}^{\leq t}$ the set of the products of length $\leq t$ and $\mathbf{e} = (1, 1, \ldots, 1)$.

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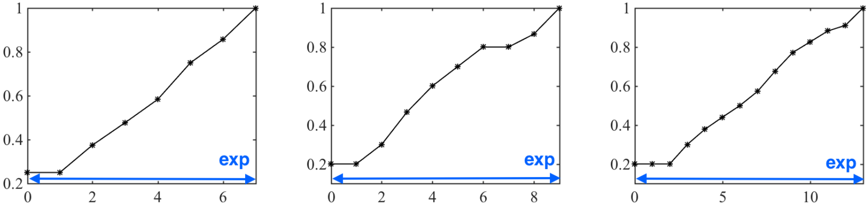

Properties

- $K_{\mathcal{M}}(t)$ is nondecreasing, $K_{\mathcal{M}}(t) = 1 \Leftrightarrow \mathcal{M}$ is primitive.
- $\min_t \{K_{\mathcal{M}}(t) = 1\} = exp(\mathcal{M})$.

It captures the **speed** at which a set reaches its first positive product

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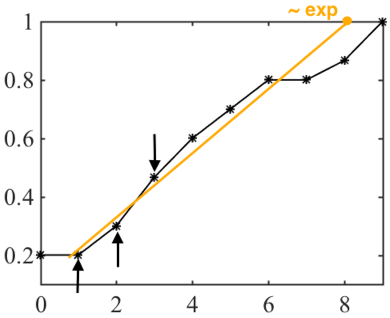

The approximated SPF: $\bar{K}(t)$

The probability that Player A wins if they both play optimally is:

$$\bar{K}_{\mathcal{M}}(t) = \min_{e_1,\dots,e_n} \left\{ \max_{M_{i_1}\cdot\ldots\cdot M_{i_l} \in \mathcal{M}^{\leq t}} {e_i}^T M_{i_1} \cdots M_{i_l} \left(\frac{\mathbf{e}}{n} \right) \right\}$$

where $\mathcal{E} = \{e_1, \dots, e_n\}$ is the canonical basis.

- $\bar{K}(t) \geq K(t)$

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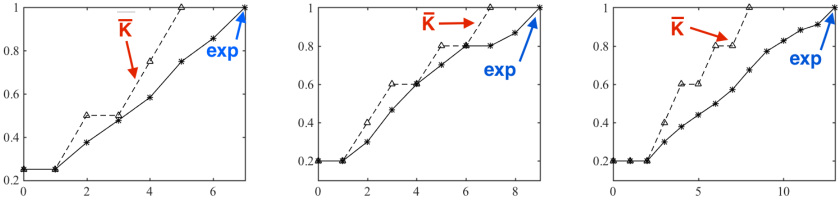


Proposition

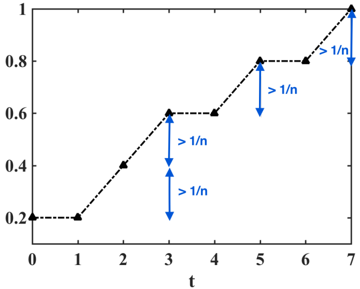
If $\min\{t : \bar{K}_{\mathcal{M}}(t) = 1\} \leq B(n)$ then $exp(\mathcal{M}) \leq 2B(n)$.

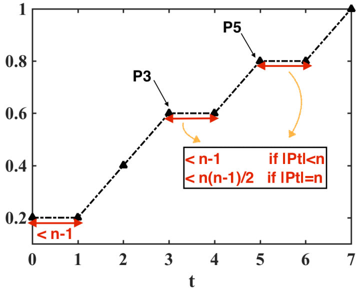
The approximated SPF: $\bar{K}(t)$

Linear programming formulation

$$\bar{K}_{\mathcal{M}}(t) = \min_{e_i,k} \frac{k}{n} \quad \text{s. t.} \begin{cases} e_i^T H_t \leq k\mathbf{e}^T \\ e_i^T \mathbf{e} = 1 \\ e_i \geq 0 \end{cases}$$

where the i-th column of H_t is the vector $A_i\mathbf{e}$ with A_i the i-th element of $\mathcal{M}^{\leq t}$. $\mathbf{P_t} \subseteq \{e_1, \ldots, e_n\}$ is the set of its **optimal solutions**.



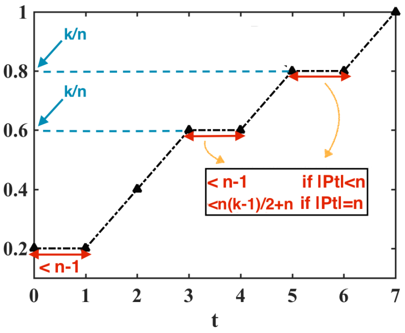


$\Rightarrow \min\{t : \bar{K}_{\mathcal{M}}(t) = 1\} \leq n(n-1)^2/2 \quad \Rightarrow exp(\mathcal{M}) \leq n(n-1)^2$ ☹

New upper bound for a class of primitive sets

Assumption

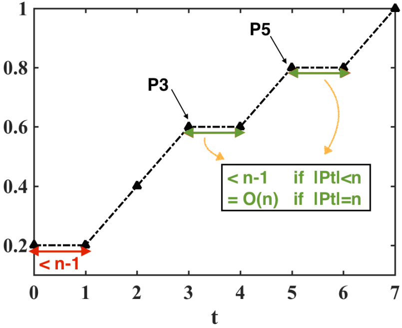
For each matrix $M_i \in \mathcal{M}$ there exists a **permutation matrix** P_i s.t. $\mathbf{M_i \geq P_i}$ entrywise.



Proposition

$exp(\mathcal{M}) \leq (n^3 - n^2 + 2n - 8)/4$ for any set $\mathcal{M}$ that fulfils the assumption.
$\longrightarrow$ **better than the known one!** but still cubic...

Future work



Conjecture

For any NZ-primitive set $\mathcal{M}$ of matrix size $n \times n$, $exp(\mathcal{M}) = O(n^2)$.

Thank you! ...questions?

C. Catalano, R. M. Jungers, *On randomized generation of slowly synchronizing automata*. Mathematical Foundations of Computer Science, 2018, 117, 48:1–48:16.

C. Catalano, R. M. Jungers, *The synchronizing probability function of primitive sets of matrices*. Developments in Language Theory, 2018, 194-205.

C. Catalano, R. M. Jungers, *Random primitive sets may generate slowly synchronizing automata*. Soon on Arxiv.