A Counterexample to Thiagarajan's Conjecture on Regular Event Structures

Jérémie Chalopin

LIS, CNRS & Aix-Marseille Université

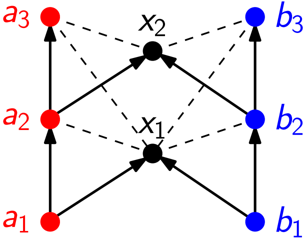
12th International Conference on Reachability Problems

Joint work with Victor Chepoi

(Prime) Event Structures

An event structure is a triple $\mathcal{E} = (E, \leq, \#)$ where

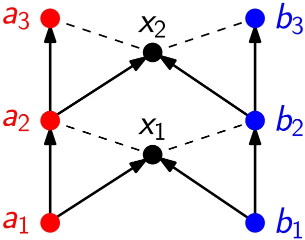
- E is a set of events
- $\leq$ is a partial order on E
- $\#$ is a (binary) conflict relation on E
- $\downarrow e := \{e' \in E : e' \leq e\}$ is finite for any $e \in E$
- $e \# e'$ and $e' \leq e'' \implies e \# e''$



(Prime) Event Structures

An event structure is a triple $\mathcal{E} = (E, \leq, \#)$ where

- E is a set of events
- $\leq$ is a partial order on E
- $\#$ is a (binary) conflict relation on E
- $\downarrow e := \{e' \in E : e' \leq e\}$ is finite for any $e \in E$
- $e \# e'$ and $e' \leq e'' \implies e \# e''$

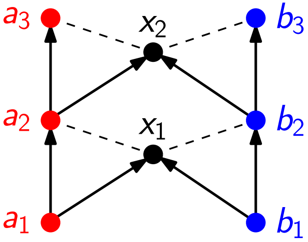


- e_1 and e_2 are in minimal conflict, $e_1 \#_\mu e_2$, if there is no event $e_1' \leq e_1$ such that $e_1' \# e_2$ (and vice versa)
- e_1 and e_2 are concurrent, $e_1 \| e_2$, if they are not comparable for $\leq$ and not in conflict

Configurations and Domains

A finite subset $c \subseteq E$ is a configuration if

- c is downward-closed: $e \in c$ and $e' \leq e \implies e' \in c$
- c is conflict-free: $e, e' \in c \implies (e, e') \notin \#$



- $\{a_1, a_2, b_1\}$ is a configuration
- $\{a_1, b_1, x_1\}$ is a configuration
- $\{a_1, a_2, b_2\}$ is not a configuration
- $\{a_1, a_2, b_1, x_1\}$ is not a configuration

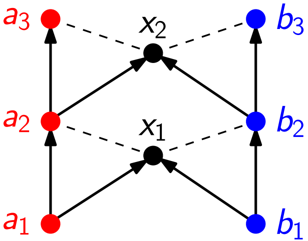
Configurations and Domains

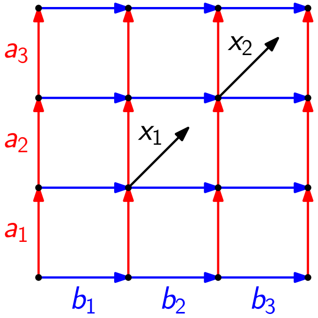
A finite subset $c \subseteq E$ is a configuration if

- c is downward-closed: $e \in c$ and $e' \leq e \implies e' \in c$
- c is conflict-free: $e, e' \in c \implies (e, e') \notin \#$

The domain $D(\mathcal{E})$ is a directed graph where

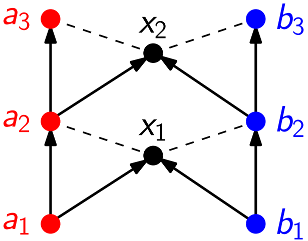
- the vertices of $D(\mathcal{E})$ are the configurations of $\mathcal{E}$
- $c \to c'$ if $c' = c \cup \{e\}$ for some event $e \notin c$

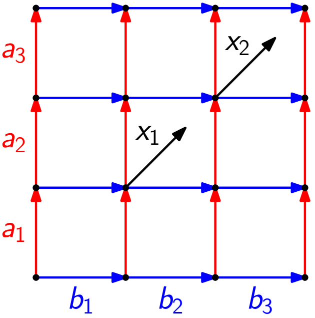




Labeled Event Structures

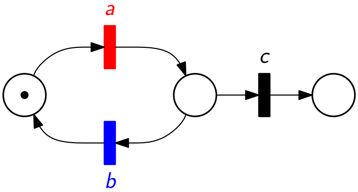
- A labeled event structure $(\mathcal{E}, \lambda)$ is an event structure $\mathcal{E}$ with a labeling $\lambda : E \to \Sigma$ (where Σ is a finite alphabet)
- λ is a nice labeling if $\lambda(e) \neq \lambda(e')$ when $e \| e'$ or $e \#_\mu e'$
- Equivalently, λ is a coloring of the edges of $D(\mathcal{E})$
  - Determinism: two edges with the same origin have distinct colors
  - Concurrency: two opposite edges of a square have the same color





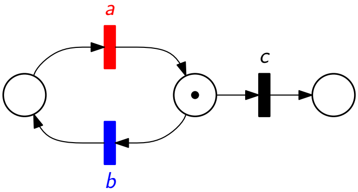
Event Structures and 1-safe Petri Nets

To any finite 1-safe Petri Net N, one can associate an event structure $\mathcal{E}_N$ with a nice labeling λ_N



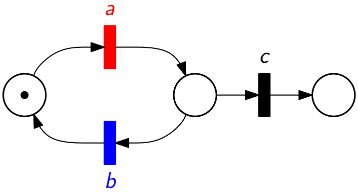
Event Structures and 1-safe Petri Nets

To any finite 1-safe Petri Net N, one can associate an event structure $\mathcal{E}_N$ with a nice labeling λ_N



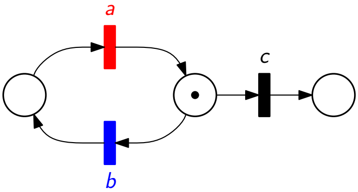
Event Structures and 1-safe Petri Nets

To any finite 1-safe Petri Net N, one can associate an event structure $\mathcal{E}_N$ with a nice labeling λ_N



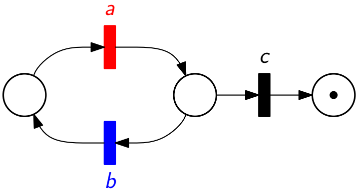
Event Structures and 1-safe Petri Nets

To any finite 1-safe Petri Net N, one can associate an event structure $\mathcal{E}_N$ with a nice labeling λ_N



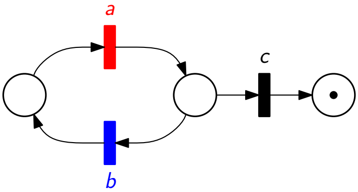
Event Structures and 1-safe Petri Nets

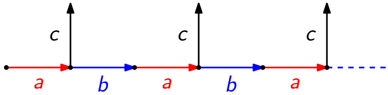
To any finite 1-safe Petri Net N, one can associate an event structure $\mathcal{E}_N$ with a nice labeling λ_N



Event Structures and 1-safe Petri Nets

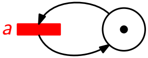
To any finite 1-safe Petri Net N, one can associate an event structure $\mathcal{E}_N$ with a nice labeling λ_N

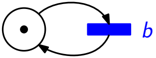


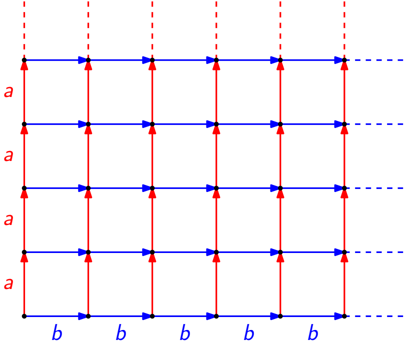


Event Structures and 1-safe Petri Nets

To any finite 1-safe Petri Net N, one can associate an event structure $\mathcal{E}_N$ with a nice labeling λ_N

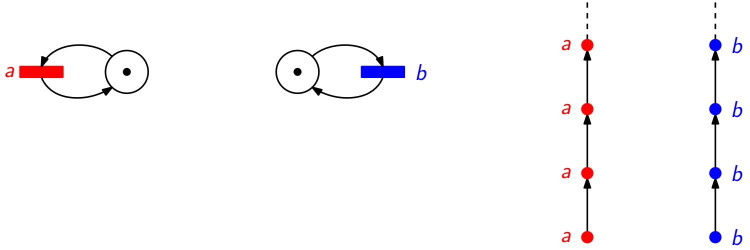






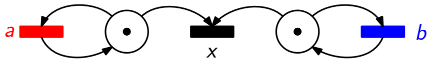
Event Structures and 1-safe Petri Nets

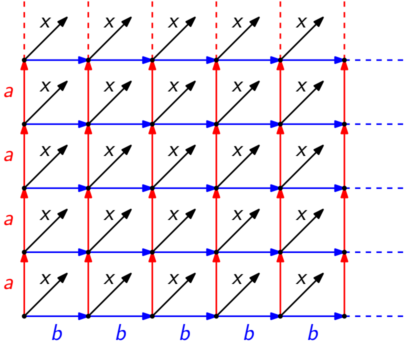
To any finite 1-safe Petri Net N, one can associate an event structure $\mathcal{E}_N$ with a nice labeling λ_N



Event Structures and 1-safe Petri Nets

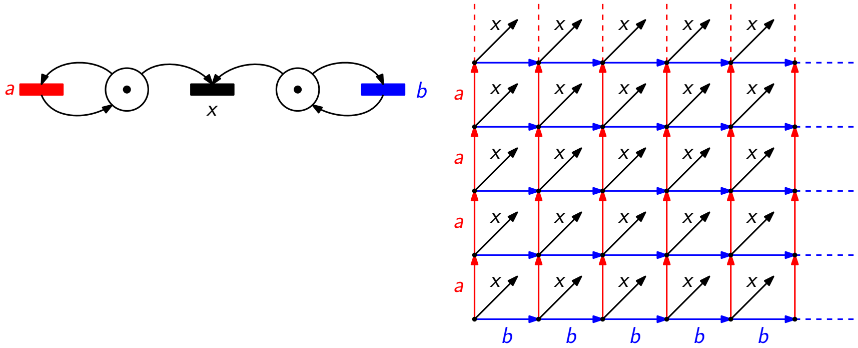
To any finite 1-safe Petri Net N, one can associate an event structure $\mathcal{E}_N$ with a nice labeling λ_N





Event Structures and 1-safe Petri Nets

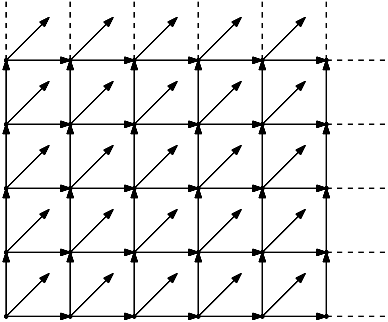
To any finite 1-safe Petri Net N, one can associate an event structure $\mathcal{E}_N$ with a nice labeling λ_N



There is some regularity in the event structures arising from 1-safe Petri Nets

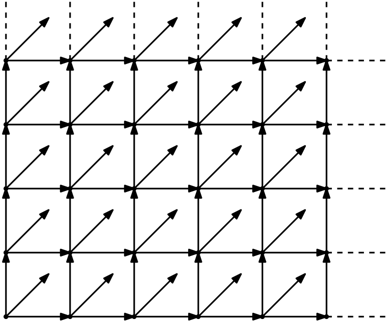
Regular Event Structures

- In $D(\mathcal{E})$, the future of a configuration c is the subgraph induced by the configurations reachable from c in $D(\mathcal{E})$
- Two configurations c, c' are equivalent, $cR_{\mathcal{E}}c'$, if they have isomorphic futures



Regular Event Structures

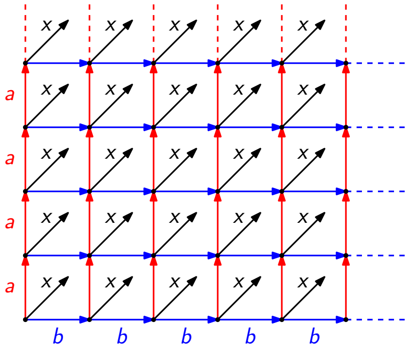
- In $D(\mathcal{E})$, the future of a configuration c is the subgraph induced by the configurations reachable from c in $D(\mathcal{E})$
- Two configurations c, c' are equivalent, $cR_{\mathcal{E}}c'$, if they have isomorphic futures
- A event structure $\mathcal{E}$ is regular if $D(\mathcal{E})$ has a finite degree and $R_{\mathcal{E}}$ has a finite number of equivalence classes



Regular Labeled Event Structures

If $(\mathcal{E}, \lambda)$ is a labeled event structure

- Two configurations c, c' are equivalent, $cR_{\mathcal{E}}c'$, if they have isomorphic labeled futures
- $(\mathcal{E}, \lambda)$ is regular if λ is a nice labeling and $R_{\mathcal{E}}$ has a finite number of equivalence classes
- We say that λ is a regular nice labeling of $\mathcal{E}$



Event Structures and 1-safe Petri Nets

Any finite 1-safe Petri net gives a regular labeled event structure (and some extra properties)

Theorem [Thiagarajan '96 (+ Morin '05)]

Any regular labeled event structure $(\mathcal{E}, \lambda)$ is isomorphic to the event structure arising from a 1-safe Petri Net

Thiagarajan's conjecture '96

Any regular event structure $\mathcal{E}$ is isomorphic to the event structure arising from a 1-safe Petri Net

- True when $\mathcal{E}$ is conflict-free [Nielsen, Thiagarajan '02]
- True when the domain of $\mathcal{E}$ is context-free [Badouel, Darondeau, Raoult '99]

Event Structures and 1-safe Petri Nets

Any finite 1-safe Petri net gives a regular labeled event structure (and some extra properties)

Theorem [Thiagarajan ’96 (+ Morin ’05)]

Any regular labeled event structure $(\mathcal{E}, \lambda)$ is isomorphic to the event structure arising from a 1-safe Petri Net

Thiagarajan’s conjecture ’96

Any regular event structure $\mathcal{E}$ is isomorphic to the event structure arising from a 1-safe Petri Net

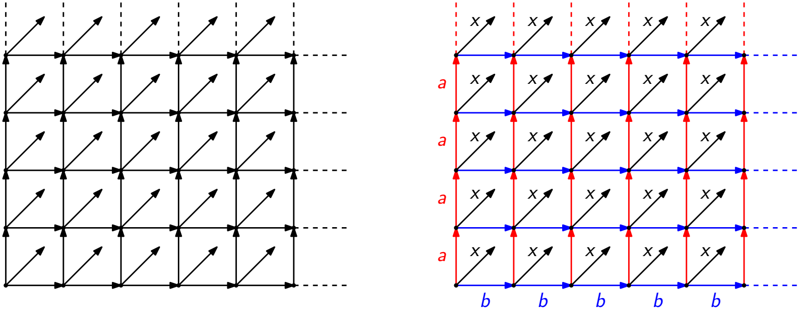
An equivalent condition

Any regular event structure $\mathcal{E}$ admits a regular nice labeling

The Problem

Our Question

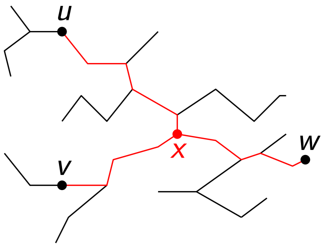
Given a regular event structure $\mathcal{E}$, can we always find a regular nice labeling of $\mathcal{E}$?



Median graphs

Definition

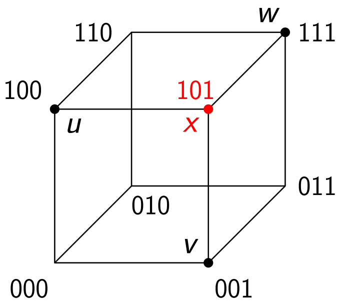
A graph $G = (V, E)$ is median if for all $u, v, w \in V$, there exists a unique $x \in V$ lying on a (u, v)-shortest path, a (u, w)-shortest path, and a (v, w)-shortest path



Median graphs

Definition

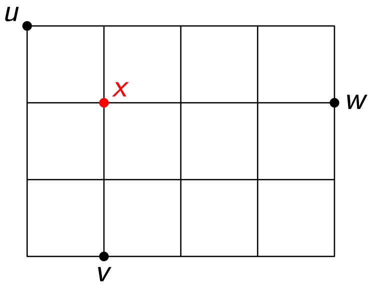
A graph $G = (V, E)$ is median if for all $u, v, w \in V$, there exists a unique $x \in V$ lying on a (u, v)-shortest path, a (u, w)-shortest path, and a (v, w)-shortest path



Median graphs

Definition

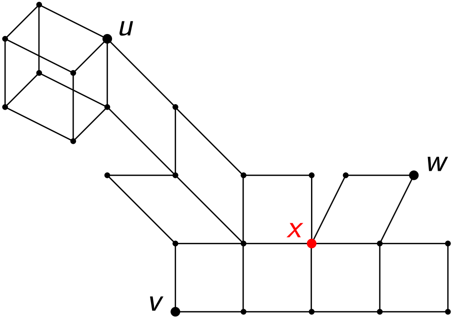
A graph $G = (V, E)$ is median if for all $u, v, w \in V$, there exists a unique $x \in V$ lying on a (u, v)-shortest path, a (u, w)-shortest path, and a (v, w)-shortest path



Median graphs

Definition

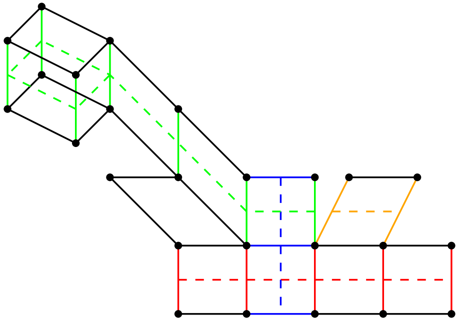
A graph $G = (V, E)$ is median if for all $u, v, w \in V$, there exists a unique $x \in V$ lying on a (u, v)-shortest path, a (u, w)-shortest path, and a (v, w)-shortest path



Hyperplanes [Sageev]

In a median graph G, the Djoković-Winkler relation Θ is defined as follows:

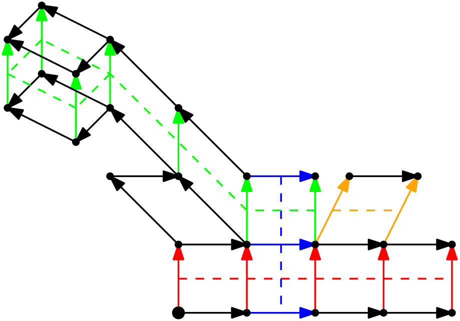
- $e_1 \Theta_1 e_2$ if e_1 and e_2 are two two opposite edges of a square
- $\Theta = \Theta_1^*$
- an hyperplane of G is an equivalence class of Θ



Median Graphs and Event Structures

Theorem [Barthélémy and Constantin '93]

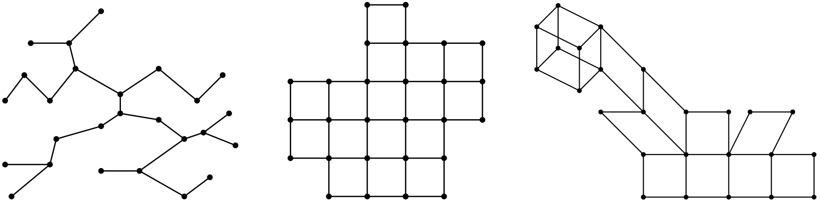
- $D(\mathcal{E})$ is a median graph (forgetting the orientation)
- Any pointed median graph is the domain of an event structure



CAT(0) cube complexes

A cube complex is a cell complex where each cell is a cube and when two cubes intersect, they intersect on a common face.

The 1-skeleton of X is the underlying graph $(V(X), E(X))$

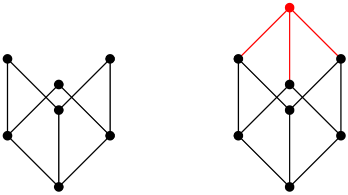


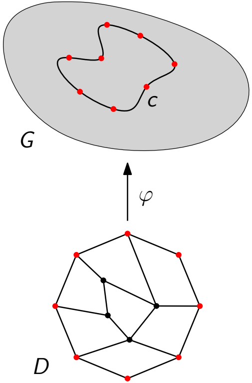
CAT(0) cube complexes

A cube complex is a cell complex where each cell is a cube and when two cubes intersect, they intersect on a common face.

A cube complex X is CAT(0) if

- X satisfies Gromov's cube condition
- X is simply connected





CAT(0) cube complexes

A cube complex is a cell complex where each cell is a cube and when two cubes intersect, they intersect on a common face.

A cube complex X is CAT(0) if

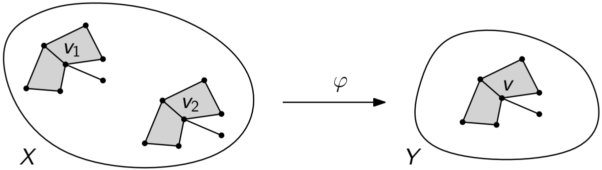
- X satisfies Gromov's cube condition
- X is simply connected

Theorem [Chepoi '00]

Median graphs are exactly the 1-skeletons of CAT(0) cube complexes

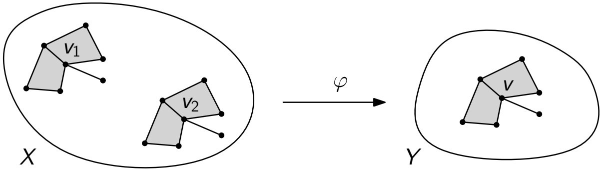
Covers of cube complexes

A cube complex X is a cover of the cube complex Y if there is a simplicial map $\varphi : V(X) \to V(Y)$ that is locally bijective



Covers of cube complexes

A cube complex X is a cover of the cube complex Y if there is a simplicial map $\varphi : V(X) \to V(Y)$ that is locally bijective



Theorem (from Topology)

- *Any complex X has a universal cover $\widetilde{X}$ such that if Y is a cover of X then $\widetilde{X}$ is a cover of Y*
- *X is simply connected if and only if $\widetilde{X} = X$*

Constructing Event Structures from NPC complexes

A cube complex is Non Positively Curved (NPC) if it satisfies Gromov's cube condition

- Starting from a finite NPC cube complex X, its universal cover $\widetilde{X}$ is a CAT(0) cube complex
- We have a finite number of equivalence classes of vertices in $\widetilde{X}$ up to isomorphism

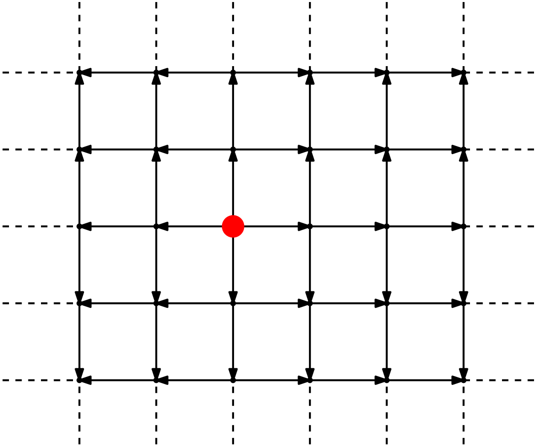
Problem

We need to have some orientations on the edges to get the domain of an event structure

Constructing Event Structures from NPC complexes

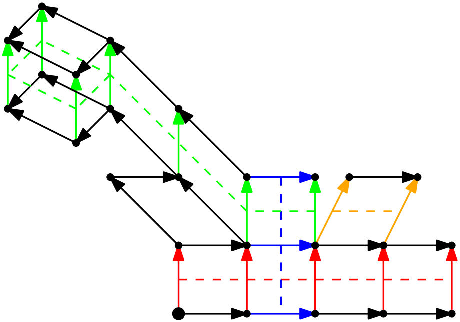
Problem

We need to have some orientations on the edges to get the domain of an event structure



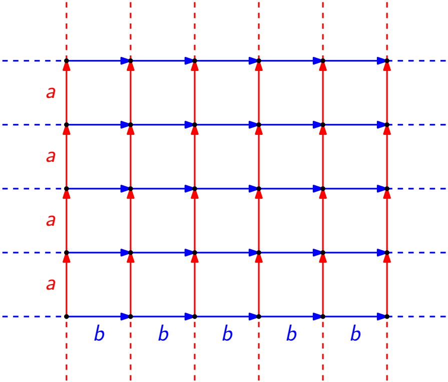
Directed NPC complexes

A directed NPC complex is a complex such that each edge is directed in such a way that two opposite edges of a square have the same direction



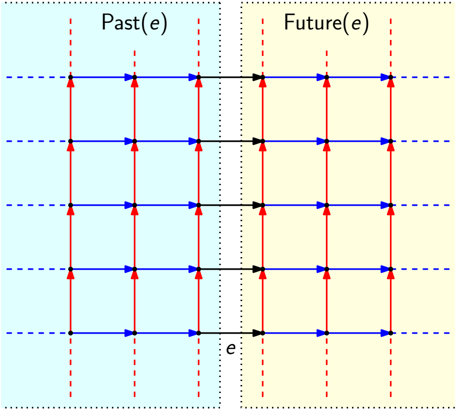
From Directed NPC complexes to Event Structures

- Starting from a finite directed NPC complex X, we construct its universal cover $\widetilde{X}$
- We have a finite number of classes of futures
- But vertices can have an infinite past ...



Cutting along Hyperplanes

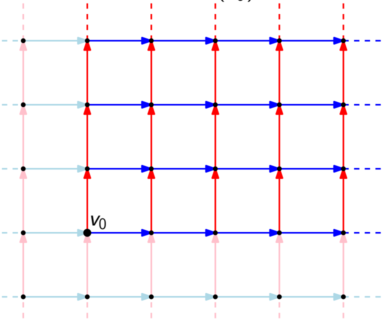
- In $\widetilde{X}$, edges belonging to the same hyperplane have the same orientation
- In a CAT(0) cube complex, hyperplanes are separators
  - For each hyperplane e, we define $\mathsf{Past}(e)$ and $\mathsf{Future}(e)$



Cutting along Hyperplanes

- In $\widetilde{X}$, edges belonging to the same hyperplane have the same orientation
- In a CAT(0) cube complex, hyperplanes are separators
- Pick $v_0 \in \widetilde{X}$, let $\mathsf{Past}(v_0) = \{e \mid v_0 \in \mathsf{Future}(e)\}$ and

$$\widetilde{X}_{v_0} = \bigcap_{e \in \mathsf{Past}(v_0)} \mathsf{Future}(e)$$



Cutting along Hyperplanes

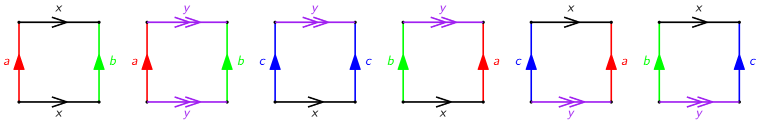
- In $\widetilde{X}$, edges belonging to the same hyperplane have the same orientation
- In a CAT(0) cube complex, hyperplanes are separators
- Pick $v_0 \in \widetilde{X}$, let $\mathsf{Past}(v_0) = \{e \mid v_0 \in \mathsf{Future}(e)\}$ and

$$\widetilde{X}_{v_0} = \bigcap_{e \in \mathsf{Past}(v_0)} \mathsf{Future}(e)$$

- Starting from a finite directed NPC complex X, we have constructed a pointed CAT(0) cube complex $\widetilde{X}_{v_0}$, i.e., the domain of an event structure
- The number of classes of futures is bounded by $|V(X)|$
- $\widetilde{X}_{v_0}$ is the domain of a regular event structure

Wise's directed NPC complex X

A colored directed NPC complex with 1 vertex, 2 "horizontal" edges (x and y), 3 "vertical" edges (a, b, and c), 6 squares



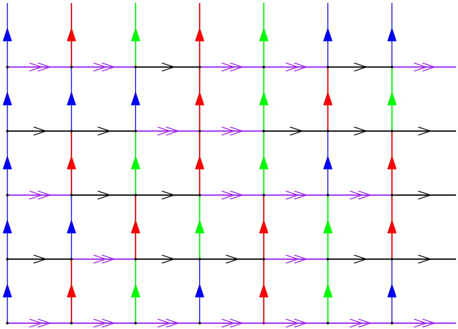
- it defines a square complex
- it is directed non positively curved

Warning!!

Colors have nothing to do with the labels of an event structure

An aperiodic tiling in the universal cover $\widetilde{X}$ of X

In the universal cover $\widetilde{X}$ of X, the quarter of plane defined by y^ω and c^ω is aperiodic

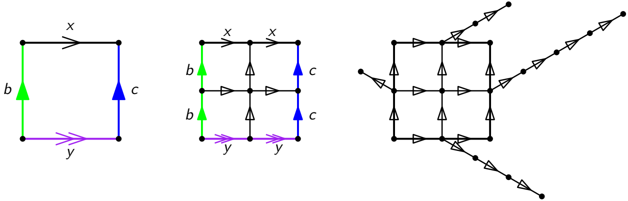


Proposition [Wise '96]

All horizontal words starting on the side of the quarter of plane are distinct

From $\widetilde{X}$ to a colorless domain $\widetilde{W}_v$

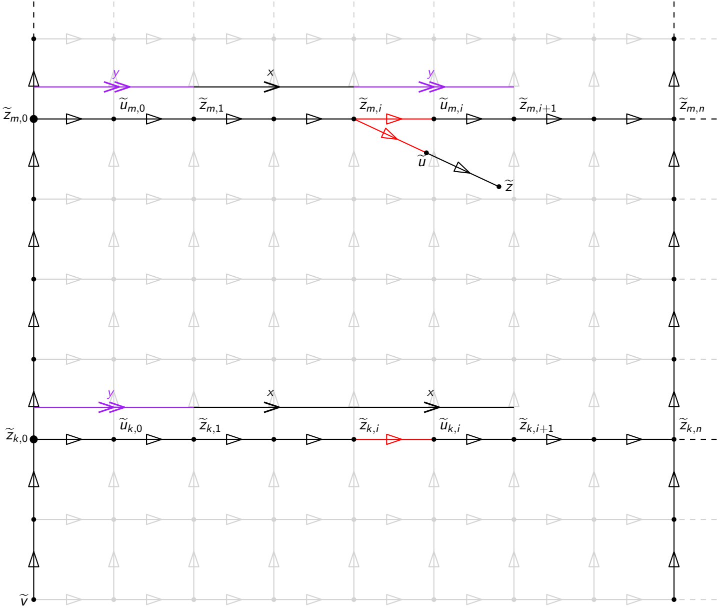
We encode the colors of the edges by a trick



In X, each color is "replaced" by a directed path attached to the "middle" of the edge

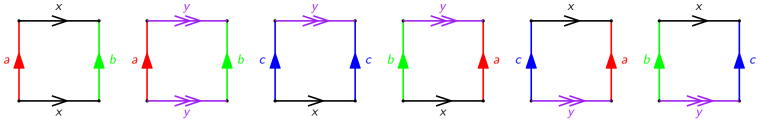
Let W be the colorless directed NPC complex obtained
Consider its universal cover $\widetilde{W}$
Pick a vertex v in $\widetilde{W}$ and consider the domain $\widetilde{W}_v$

$\widetilde{W}_v$ has no regular nice labeling



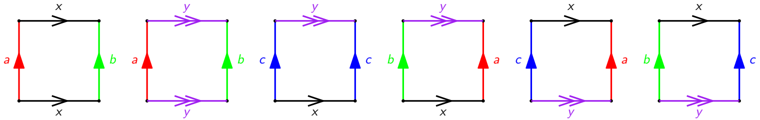
Counterexamples arise from aperiodic tilesets

Wise's complex is obtained from a 4-way deterministic tileset



Counterexamples arise from aperiodic tilesets

Wise's complex is obtained from a 4-way deterministic tileset



Any aperiodic 4-way deterministic tileset gives a counterexample to Thiagarajan's conjecture

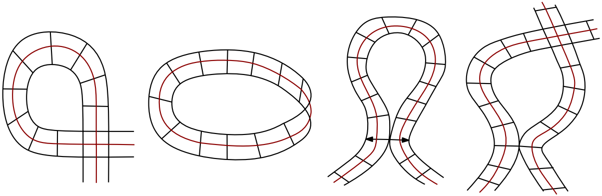
Theorem

- There exists a 4-way deterministic aperiodic tileset [Kari, Papasoglu '99]
- Deciding if a 4-way deterministic tileset tiles the plane is undecidable [Lukkarila '09]

On the positive side: special cube complexes

A NPC complex is special if its hyperplanes behave nicely

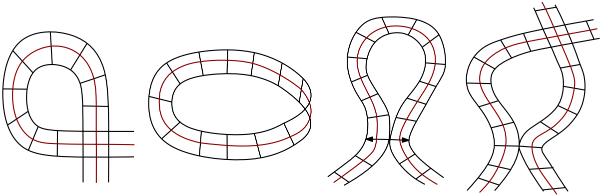
[Haglund, Wise '08]



A finite NPC complex is virtually special if it has a finite cover that is special

On the positive side: special cube complexes

A NPC complex is special if its hyperplanes behave nicely [Haglund, Wise '08]



A finite NPC complex is virtually special if it has a finite cover that is special

Theorem [Agol'13]

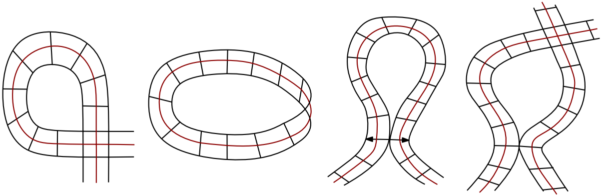
If the universal cover $\widetilde{X}$ of a finite NPC complex X is hyperbolic, then X is virtually special

$\widetilde{X}$ is hyperbolic $\Leftrightarrow$ isometric square grids in $\widetilde{X}$ are bounded

On the positive side: special cube complexes

A NPC complex is special if its hyperplanes behave nicely

[Haglund, Wise '08]



A finite NPC complex is virtually special if it has a finite cover that is special

Theorem

A finite NPC complex X is special iff for any orientation of X, Thiagarajan's conjecture is true for all $\widetilde{X}_v$ (this is the case in particular if $\widetilde{X}$ is hyperbolic)

1-safe Petri nets and special cube complexes

Theorem

An event structure $\mathcal{E}$ admits a regular nice labeling

$\Leftrightarrow$ *$\mathcal{E}$ is isomorphic to the event structure arising from a 1-safe Petri Net* *[Thiagarajan '96]*

$\Leftrightarrow$ *there exists a finite directed (virtually) special cube complex X such that $D(\mathcal{E}) \simeq \widetilde{X}_v$*

1-safe Petri nets and special cube complexes

Theorem

An event structure $\mathcal{E}$ admits a regular nice labeling

$\Leftrightarrow$ *$\mathcal{E}$ is isomorphic to the event structure arising from a 1-safe Petri Net* *[Thiagarajan '96]*

$\Leftrightarrow$ *there exists a finite directed (virtually) special cube complex X such that $D(\mathcal{E}) \simeq \widetilde{X}_v$*

Question

Can any regular event domain be obtained from the universal cover of a finite NPC complex?

If the answer is negative, it will give new counterexamples to Thiagarajan's conjecture

Conclusion

- A negative result,
  - A counter-example to Thiagarajan's conjecture
  - In fact, many counter-examples arising from 4-way deterministic aperiodic tilesets

Conclusion

- A negative result,
  - A counter-example to Thiagarajan's conjecture
  - In fact, many counter-examples arising from 4-way deterministic aperiodic tilesets
- On the positive side, the conjecture is true for particular ("antinomic") cases
  - conflict-free event structures [Nielsen, Thiagarajan '02]
  - context-free event domains [Badouel, Darondeau, Raoult '99]
  - domains obtained from finite NPC complexes with an hyperbolic universal cover

Conclusion

- A negative result,
  - A counter-example to Thiagarajan's conjecture
  - In fact, many counter-examples arising from 4-way deterministic aperiodic tilesets
- On the positive side, the conjecture is true for particular ("antinomic") cases
  - conflict-free event structures [Nielsen, Thiagarajan '02]
  - context-free event domains [Badouel, Darondeau, Raoult '99]
  - domains obtained from finite NPC complexes with an hyperbolic universal cover
- Questions:
  - Is Thiagarajan's conjecture true for hyperbolic domains?
  - Can we decide if a regular event structure admits a regular nice labelling?

Conclusion

- Nice connections between event structures and NPC complexes
  - CAT(0) cube complexes correspond to event structures
  - finite (virtually) special cube complexes correspond to regular event structures that admit a regular nice labeling
  - Question: Do finite NPC complexes correspond to regular event structures?

Conclusion

- Nice connections between event structures and NPC complexes
  - CAT(0) cube complexes correspond to event structures
  - finite (virtually) special cube complexes correspond to regular event structures that admit a regular nice labeling
  - Question: Do finite NPC complexes correspond to regular event structures?

Thank you! Questions?