Qualitative Reachability for Open Interval Markov Chains

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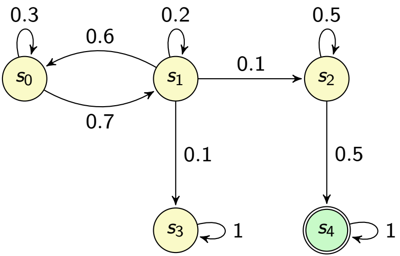
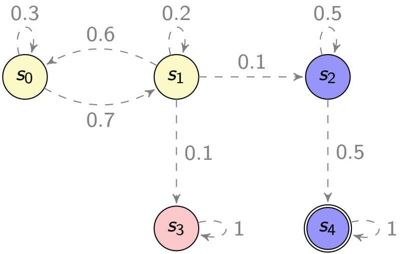
Qualitative reachability in Markov chains

Input:
– Markov chain
– Target states T

Output:
– $S^{=0}$: set of states reaching T with probability 0
– $S^{=1}$: set of states reaching T with probability 1

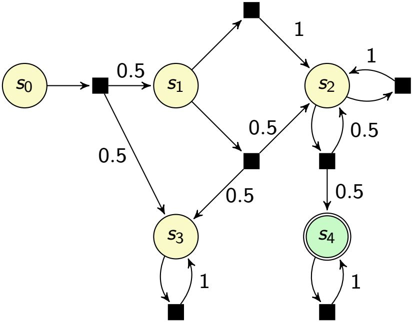


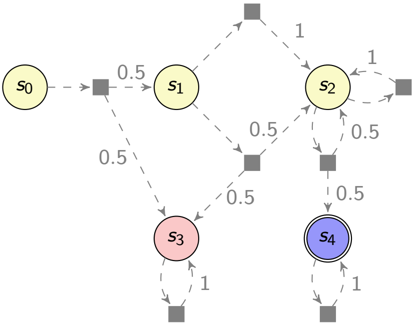


Qualitative reachability in Markov decision processes

Input:
– Markov decision process (MDP): transition consists of nondeterministic choice of distribution from source state, then probabilistic choice of target state according to the distribution
– Target states T

Output:
– $S_{\forall}^{=0}$: set of states reaching T with probability 0 for all schedulers (resolutions of nondeterminism)
– $S_{\forall}^{=1}$: set of states reaching T with probability 1 for all schedulers (resolutions of nondeterminism)

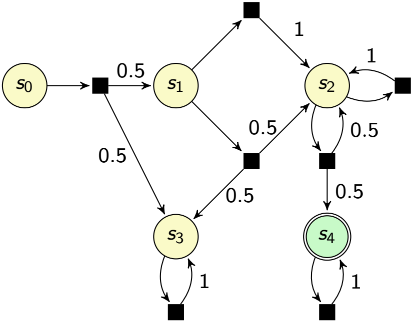


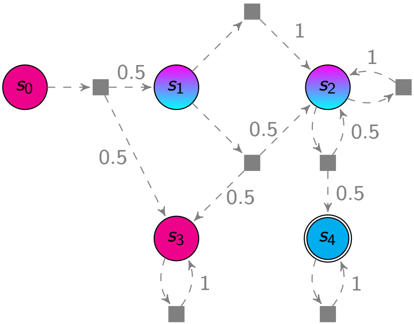


Qualitative reachability in Markov decision processes

Input:
– Markov decision process (MDP): transition consists of nondeterministic choice of distribution from source state, then probabilistic choice of target state according to the distribution
– Target states T

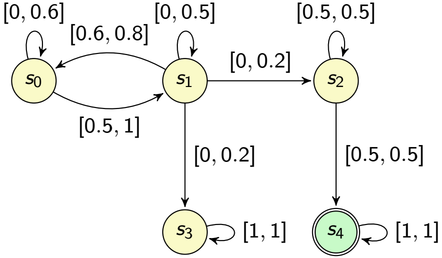
Output:
– $S_{\exists}^{=0}$: set of states reaching T with probability 0 for some scheduler (resolution of nondeterminism)
– $S_{\exists}^{=1}$: set of states reaching T with probability 1 for some scheduler (resolution of nondeterminism)





Interval Markov chains

- Precise information regarding transition probabilities may not be available.
- *Interval Markov chains* (IMCs): Markov chains where transition probabilities are replaced by *intervals* [JL91,KU02].

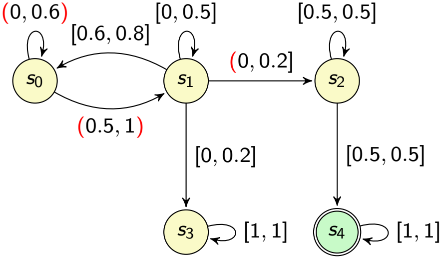


[JL91] B. Jonsson and K. G. Larsen. Specification and refinement of probabilistic processes. In *Proc. LICS 1991*.

[KU02] I. O. Kozine and L. V. Utkin. Interval-valued finite Markov chains. *Reliable Computing*, 8(2), 2002.

Open interval Markov chains

- Open IMCs: use (half-)open intervals, in addition to closed intervals [CK15].



[CK15]. S. Chakraborty and J.-P. Katoen. Model Checking of Open Interval Markov Chains. In *Proc. ASMTA 2015*.

Reachability in IMCs

- For *closed IMCs*, quantitative reachability (more general than qualitative reachability) can be decided in *polynomial time* [CHK13,PLSS13].
- For *open IMCs*, quantitative reachability probabilities can be *approximated*.
  - Transforming an open IMC to a closed IMC by closing all intervals labelling transitions gives an arbitrarily close approximation [CK15].
- What about *exact* verification of *qualitative* reachability probabilities in open IMCs?

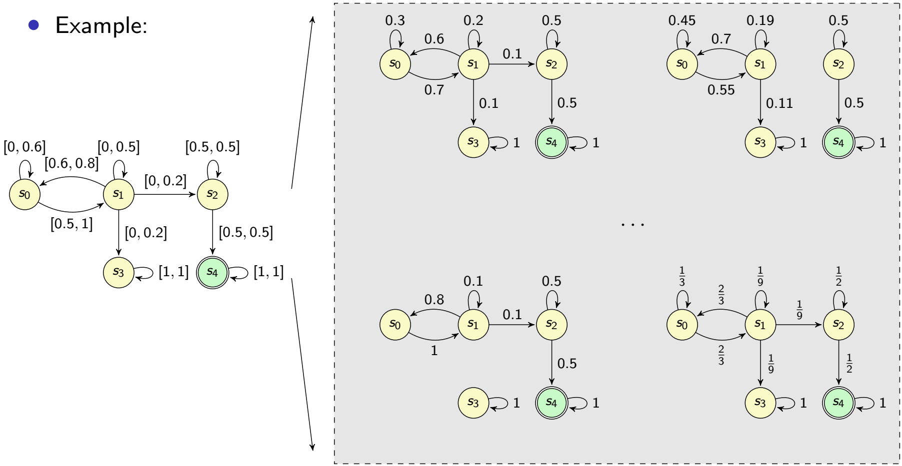
[CHK13]. T. Chen, T. Han, M. Kwiatkowska. On the Complexity of Model Checking Interval-valued Discrete Time Markov Chains. *Information Processing Letters*, 113(7), 2013.

[CK15]. S. Chakraborty and J.-P. Katoen. Model Checking of Open Interval Markov Chains. In *Proc. ASMTA 2015*.

[PLSS13]. A. Puggelli *et al.* Polynomial-Time Verification of PCTL Properties of MDPs with Convex Uncertainties. In *Proc. CAV 2013*.

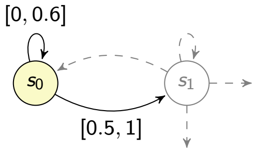
Uncertain Markov chain semantics of an IMC

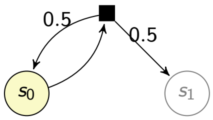
- The *uncertain Markov chain* (UMC) semantics is an (uncountable) set of Markov chains.
  - Each Markov chain corresponds to a certain choice of probabilities from the intervals of each transition.
- Example:



Interval Markov decision process semantics of an IMC

- The *interval Markov decision process* (IMDP) semantics is an MDP with an (uncountable) number of transition distributions.
  - The probabilities of each distribution associated with state s corresponds to a certain choice of probabilities from the intervals of the outgoing edges of s.
- Example: state s_0





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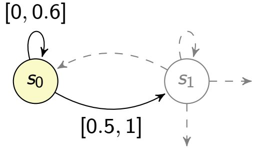

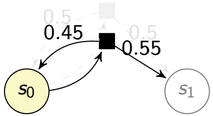

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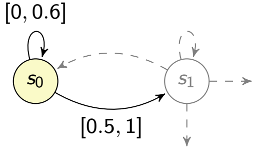

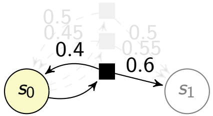

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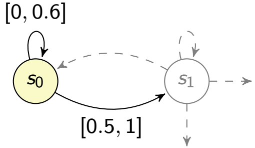

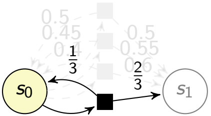

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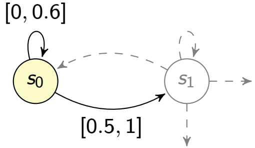

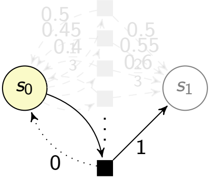

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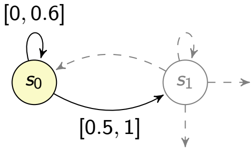
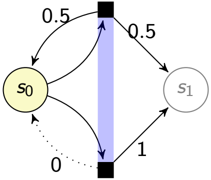
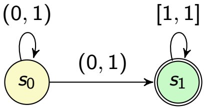




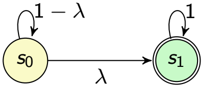
Uncountable set of distributions available in s_0:
between $\{s_0 \mapsto 0.5, s_1 \mapsto 0.5\}$ and $\{s_0 \mapsto 0, s_1 \mapsto 1\}$

UMC semantics vs. IMDP semantics: example

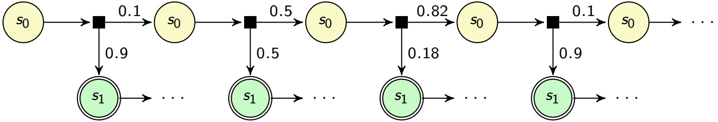
- Example IMC:



- UMC semantics: for each $\lambda \in (0,1)$, $[\![\mathcal{O}]\!]_{\mathrm{UMC}}$ contains a Markov chain $\mathcal{D}_\lambda$ of the form:



- IMDP semantics: on each visit to s_0, can choose a different distribution, for example:



Qualitative reachability in interval Markov chains

- $S_{\forall}^{0,\mathrm{UMC}} = \{s \in S \mid \forall \mathcal{D} \in [\![\mathcal{O}]\!]_{\mathrm{UMC}} \,.\, \Pr_s^{\mathcal{D}}(\mathsf{Reach}(T)) = 0\}$;
- $S_{\exists}^{0,\mathrm{UMC}} = \{s \in S \mid \exists \mathcal{D} \in [\![\mathcal{O}]\!]_{\mathrm{UMC}} \,.\, \Pr_s^{\mathcal{D}}(\mathsf{Reach}(T)) = 0\}$;
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- $S_{\forall}^{0,\mathrm{IMDP}} = \{s \in S \mid \forall \sigma \in \mathsf{Sched}^{[\![\mathcal{O}]\!]_{\mathrm{IMDP}}} \,.\, \Pr_s^{\sigma}(\mathsf{Reach}(T)) = 0\}$;
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(where $[\![\mathcal{O}]\!]_{\mathrm{UMC}}$ is the UMC semantics of IMC $\mathcal{O}$, and $\mathsf{Sched}^{[\![\mathcal{O}]\!]_{\mathrm{IMDP}}}$ is the set of schedulers of $[\![\mathcal{O}]\!]_{\mathrm{IMDP}}$).

Main results

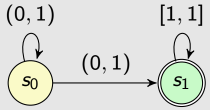
UMC and IMDP semantics coincide for all cases bar "universal/probability 1"

$S_{\forall}^{0,\mathrm{UMC}} = S_{\forall}^{0,\mathrm{IMDP}}$, $S_{\exists}^{0,\mathrm{UMC}} = S_{\exists}^{0,\mathrm{IMDP}}$ and $S_{\exists}^{1,\mathrm{UMC}} = S_{\exists}^{1,\mathrm{IMDP}}$.
There exists an open IMC such that $S_{\forall}^{1,\mathrm{UMC}} \neq S_{\forall}^{1,\mathrm{IMDP}}$.

Qualitative reachability for open IMCs can be decided efficiently

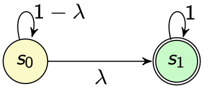
$S_{\forall}^{0,\mathrm{UMC}}$, $S_{\exists}^{0,\mathrm{UMC}}$, $S_{\exists}^{1,\mathrm{UMC}}$, $S_{\forall}^{1,\mathrm{UMC}}$, $S_{\forall}^{0,\mathrm{IMDP}}$, $S_{\exists}^{0,\mathrm{IMDP}}$, $S_{\exists}^{1,\mathrm{IMDP}}$ and $S_{\forall}^{1,\mathrm{IMDP}}$ can be computed in polynomial time in the size of the IMC.

Open IMC witnessing $S_\forall^{1,\mathrm{UMC}} \neq S_\forall^{1,\mathrm{IMDP}}$



$s_0 \in S_\forall^{1,\mathrm{UMC}}$ but $s_0 \notin S_\forall^{1,\mathrm{IMDP}}$

- $s_0 \in S_\forall^{1,\mathrm{UMC}}$: recall that all Markov chains $\mathcal{D}_\lambda$ in $[\![\mathcal{O}]\!]_{\mathrm{UMC}}$ are of the form



  for $\lambda \in (0,1)$, hence $\mathrm{Pr}_{s_0}^{\mathcal{D}_\lambda}(\mathsf{Reach}(\{s_1\})) = \lim_{k\to\infty} 1-(1-\lambda)^k = 1$.

- $s_0 \notin S_\forall^{1,\mathrm{IMDP}}$:
  - Consider scheduler (with memory) σ that assigns $\frac{1}{2^i}$ probability to the i-th attempt to take the transition from s_0 to s_1.
  - $\mathrm{Pr}_{s_0}^{\sigma}(\mathsf{Reach}(\{s_1\})) = \frac{1}{2} + \frac{1}{2}(\frac{1}{4} + \frac{3}{4}(\frac{1}{8} + \cdots)) < 1$.

Qualitative reachability in interval Markov chains

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Universal quantification/probability 0

- Complement sets:
  - $S \setminus S_{\forall}^{0,\mathrm{UMC}} = \{s \in S \mid \exists \mathcal{D} \in [\![\mathcal{O}]\!]_{\mathrm{UMC}} \,.\, \mathrm{Pr}_s^{\mathcal{D}}(\mathsf{Reach}(T)) > 0\}$.
  - $S \setminus S_{\forall}^{0,\mathrm{IMDP}} = \{s \in S \mid \exists \sigma \in \mathsf{Sched}^{[\![\mathcal{O}]\!]_{\mathrm{IMDP}}} \,.\, \mathrm{Pr}_s^{\sigma}(\mathsf{Reach}(T)) > 0\}$.

Computation of $S \setminus S_{\forall}^{0,\mathrm{UMC}}$ and $S \setminus S_{\forall}^{0,\mathrm{IMDP}}$ by graph reachability

- $s \in S \setminus S_{\forall}^{0,\mathrm{UMC}}$ iff there exists a path in the graph of the IMC from s to T.
- $s \in S \setminus S_{\forall}^{0,\mathrm{IMDP}}$ iff there exists a path in the graph of the IMC from s to T.

- Consequence: $S_{\forall}^{0,\mathrm{UMC}} = S_{\forall}^{0,\mathrm{IMDP}}$; can be computed in polynomial time.

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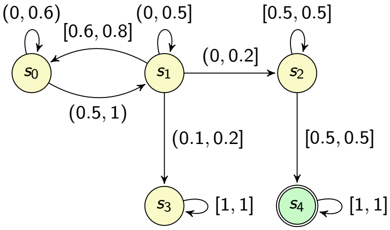
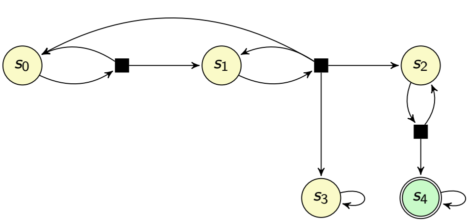
Existential quantification/probability 0 and 1

- Valid edge set: set of edges with
  1. Same source state;
  2. At least one assignment of positive probabilities to edges that respects the edges' intervals.
- Qualitative MDP abstraction: represent each valid edge set by an (arbitrary) *representative distribution* over the set's edges.
  - Justification: in finite MDPs, exact (positive) probability values are immaterial for qualitative reachability properties.
- Example:





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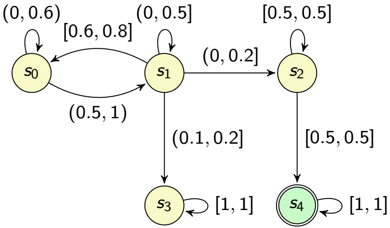
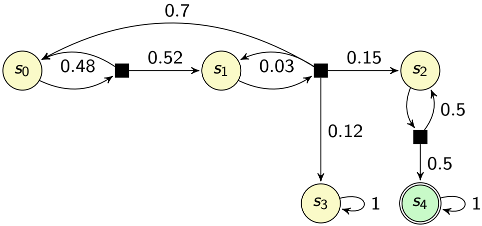
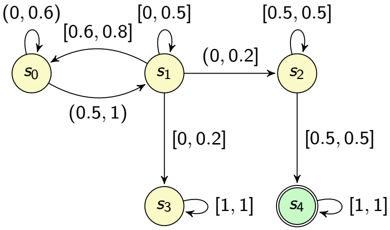
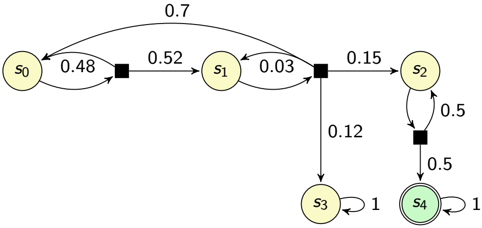




Existential quantification/probability 0 and 1

- Example with eliminable edges (edges with left-closed intervals for which the left endpoint equals 0):





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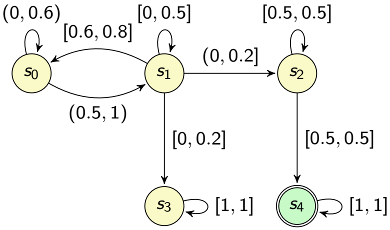

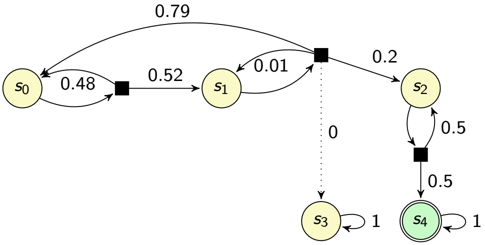

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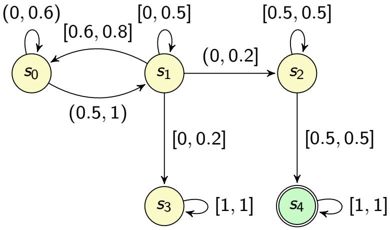

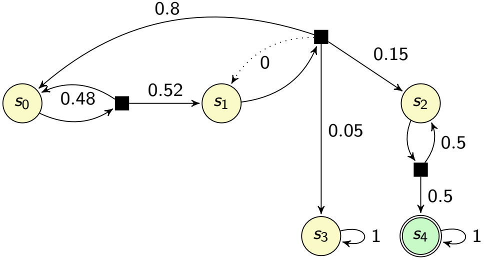

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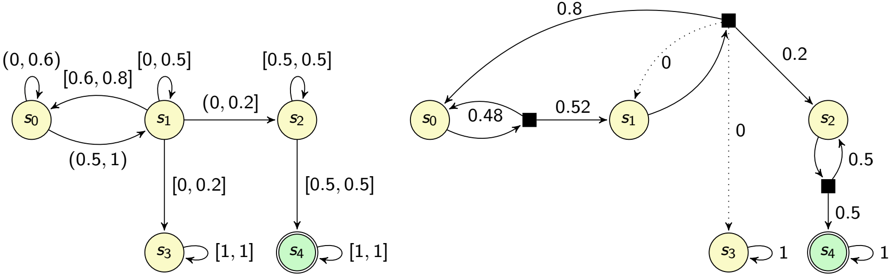

Existential quantification/probability 0 and 1

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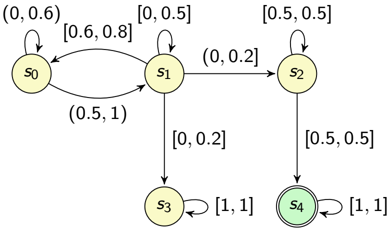

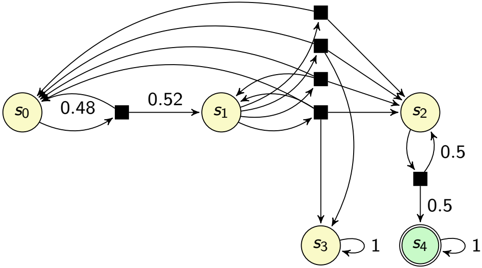

Existential quantification/probability 0 and 1

Preservation of existential qualitative reachability

Let $\lambda \in \{0, 1\}$. There exists a scheduler $\sigma \in \mathsf{Sched}^{\mathsf{QMA}(\mathcal{O})}$ of the qualitative MDP abstraction such that $\mathrm{Pr}_s^{\sigma}(\mathsf{Reach}(T)) = \lambda$ if and only if:

- there exists $\mathcal{D} \in [\![\mathcal{O}]\!]_{\mathrm{UMC}}$ such that $\mathrm{Pr}_s^{\mathcal{D}}(\mathsf{Reach}(T)) = \lambda$;
- there exists $\sigma' \in \mathsf{Sched}^{[\![\mathcal{O}]\!]_{\mathrm{IMDP}}}$ such that $\mathrm{Pr}_s^{\sigma'}(\mathsf{Reach}(T)) = \lambda$.

- Hence:
  - $S_{\exists}^{\lambda,\mathrm{UMC}} = \{s \in S \mid \exists \sigma \in \mathsf{Sched}^{\mathsf{QMA}(\mathcal{O})} . \mathrm{Pr}_s^{\sigma}(\mathsf{Reach}(T)) = \lambda\}$;
  - $S_{\exists}^{\lambda,\mathrm{IMDP}} = \{s \in S \mid \exists \sigma \in \mathsf{Sched}^{\mathsf{QMA}(\mathcal{O})} . \mathrm{Pr}_s^{\sigma}(\mathsf{Reach}(T)) = \lambda\}$.
- Proposal: compute $\{s \in S \mid \exists \sigma \in \mathsf{Sched}^{\mathsf{QMA}(\mathcal{O})} . \mathrm{Pr}_s^{\sigma}(\mathsf{Reach}(T)) = \lambda\}$ by analysing the qualitative MDP abstraction using established techniques for qualitative reachability properties of finite MDPs.

Existential quantification/probability 0 and 1

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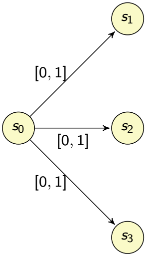
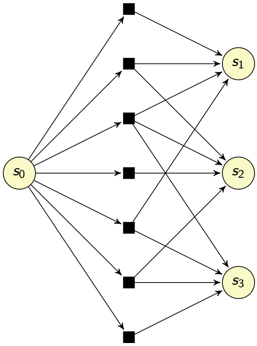
- But ...

Existential quantification/probability 0 and 1

- Size of the qualitative MDP abstraction: exponential in the size of the IMC.
  - If an IMC state has n eliminable edges, the corresponding state of the qualitative MDP abstraction has in the worst case $2^n - 1$ outgoing distributions (depends on the endpoints of the intervals).
  - Example with $n = 3$:





Existential quantification/probability 0 and 1

- Obtain algorithms applied to the qualitative MDP abstraction that run in polynomial time *in the size of the IMC*?
  - Sufficient to compute predecessor operations in polynomial time in the size of the IMC (precedent in the quantitative context in [HM18]).
- Example of predecessor operation: CPre (used for probability 0 case).
  - For $X \subseteq S$, $\mathrm{CPre}(X)$ is the set of states for which there exists a distribution such that all of the distributions edges lead to states in X.
  - Formally, $\mathrm{CPre}(X) = \{s \in S \mid \exists \mu \in \Delta_{\mathrm{QMA}}(s) \,.\, \mathrm{support}(\mu) \subseteq X\}$, where $\Delta_{\mathrm{QMA}}(s)$ is the set of distributions available in state s in $\mathrm{QMA}(\mathcal{O})$.

$\mathrm{CPre}(X)$ can be computed in polynomial time in the size of the IMC

Given $s \in S$, we have $s \in \mathrm{CPre}(X)$ if and only if

1. edges from s to $S \setminus X$ are eliminable, and
2. the sum of the right endpoints of edges from s to X is at least 1 (strictly greater than 1 if at least one edge from s to X is right open).

[HM18]. B. Monmege and S. Haddad. Interval iteration algorithm for MDPs and IMDPs. *Theoretical Computer Science*, 735, 2018.

Existential quantification/probability 0 and 1

- Recall: $s \in \mathrm{CPre}(X)$ if and only if
  1. edges from s to $S \setminus X$ are eliminable, and
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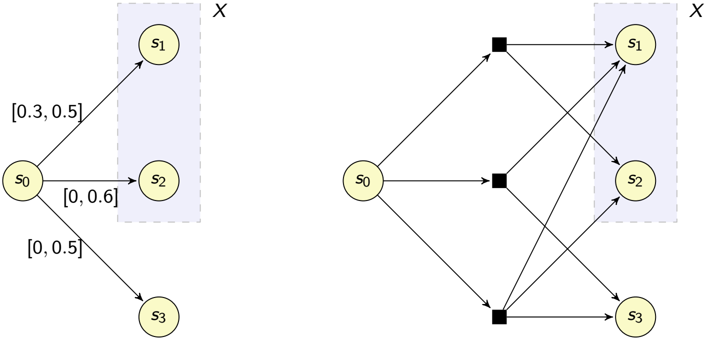

Existential quantification/probability 0 and 1

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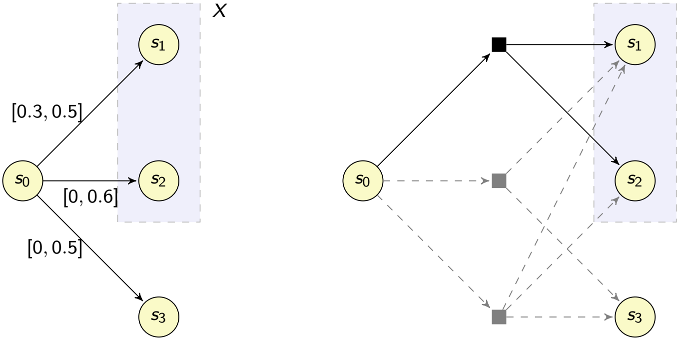

Qualitative reachability in interval Markov chains

- $S_{\forall}^{0,\mathrm{UMC}} = \{s \in S \mid \forall \mathcal{D} \in [\![\mathcal{O}]\!]_{\mathrm{UMC}} \,.\, \mathrm{Pr}_s^{\mathcal{D}}(\mathsf{Reach}(T)) = 0\}$;
- $S_{\exists}^{0,\mathrm{UMC}} = \{s \in S \mid \exists \mathcal{D} \in [\![\mathcal{O}]\!]_{\mathrm{UMC}} \,.\, \mathrm{Pr}_s^{\mathcal{D}}(\mathsf{Reach}(T)) = 0\}$;
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Universal quantification/probability 1: UMC semantics

- Aim: computation of $S_{\forall}^{1,\mathrm{UMC}} = \{s \in S \mid \forall \mathcal{D} \in [\![\mathcal{O}]\!]_{\mathrm{UMC}} \,.\, \mathrm{Pr}_s^{\mathcal{D}}(\mathsf{Reach}(T)) = 1\}$.
- Compute $S \setminus S_{\forall}^{1,\mathrm{UMC}}$, i.e., $\{s \in S \mid \exists \mathcal{D} \in [\![\mathcal{O}]\!]_{\mathrm{UMC}} \,.\, \mathrm{Pr}_s^{\mathcal{D}}(\mathsf{Reach}(T)) < 1\}$.

Universal quantification/probability 1: UMC semantics

There exists $\mathcal{D} \in [\![\mathcal{O}]\!]_{\mathrm{UMC}}$ such that $\mathrm{Pr}_s^{\mathcal{D}}(\mathsf{Reach}(T)) < 1$ if and only if there exists a finite path of the graph of $\mathcal{O}$ from s to a state in $S_{\exists}^{0,\mathrm{UMC}}$.

- Algorithm for computing $S_{\forall}^{1,\mathrm{UMC}}$: take the complement of the set of states that can reach $S_{\exists}^{0,\mathrm{UMC}}$ in the graph of the IMC.

Qualitative reachability in interval Markov chains

- $S_{\forall}^{0,\mathrm{UMC}} = \{s \in S \mid \forall \mathcal{D} \in [\![\mathcal{O}]\!]_{\mathrm{UMC}} \,.\, \mathrm{Pr}_s^{\mathcal{D}}(\mathsf{Reach}(T)) = 0\}$;
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Universal quantification/probability 1: IMDP semantics

- Aim: computation of $S_{\forall}^{1,\mathrm{IMDP}} = \{s \in S \mid \forall \sigma \in \mathsf{Sched}^{[\![\mathcal{O}]\!]_{\mathrm{IMDP}}} . \Pr_s^{\sigma}(\mathsf{Reach}(T)) = 1\}$.
- Compute $S \setminus S_{\forall}^{1,\mathrm{IMDP}}$, i.e., $\{s \in S \mid \exists \sigma \in \mathsf{Sched}^{[\![\mathcal{O}]\!]_{\mathrm{IMDP}}} . \Pr_s^{\sigma}(\mathsf{Reach}(T)) < 1\}$.
- $U_{\neg T}$: state set, not including any states in T, in which the IMC can confine itself with positive probability (defined on the next slide).

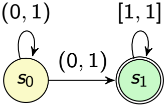
Universal quantification/probability 1: IMDP semantics

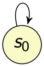
There exists $\sigma \in \mathsf{Sched}^{[\![\mathcal{O}]\!]_{\mathrm{IMDP}}}$ such that $\Pr_s^{\sigma}(\mathsf{Reach}(T)) < 1$ if and only if there exists a finite path of the graph of $\mathcal{O}$ from s to a state in $U_{\neg T}$.

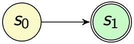
- Algorithm for computing $S_{\forall}^{1,\mathrm{IMDP}}$: take the complement of the set of states that can reach $U_{\neg T}$ in the graph of the IMC.

Universal quantification/probability 1: IMDP semantics

- Characterisation and computation of the set $U_{\neg T}$?
- Strongly-connected components with no state in T for which the probability of outgoing edges can be made arbitrarily small (requires left endpoint 0, but can be left-open).
- Example:



- $U_{\neg T} = \{s_0\}$, because 

is a strongly-connected component, and the probability of 

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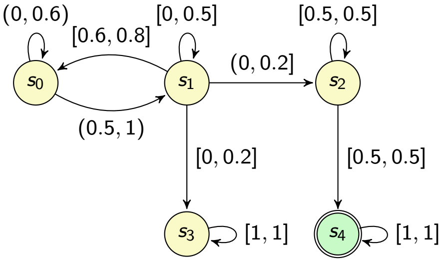
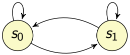
Universal quantification/probability 1: IMDP semantics

- Example:



- $U_{\neg T} = \{s_0, s_1\}$, because 

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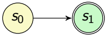
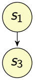
  and the probability of 

and 

can be made arbitrarily small.

Conclusions

- Exact and efficient computation of state sets satisfying qualitative reachability properties for open IMCs.
- Future work:
  - Qualitative ω-regular properties.
  - Exact computation of state sets satisfying *quantitative* reachability properties for open IMCs.