Reachability Analysis of Nonlinear ODEs using Polytopic Based Validated Runge-Kutta

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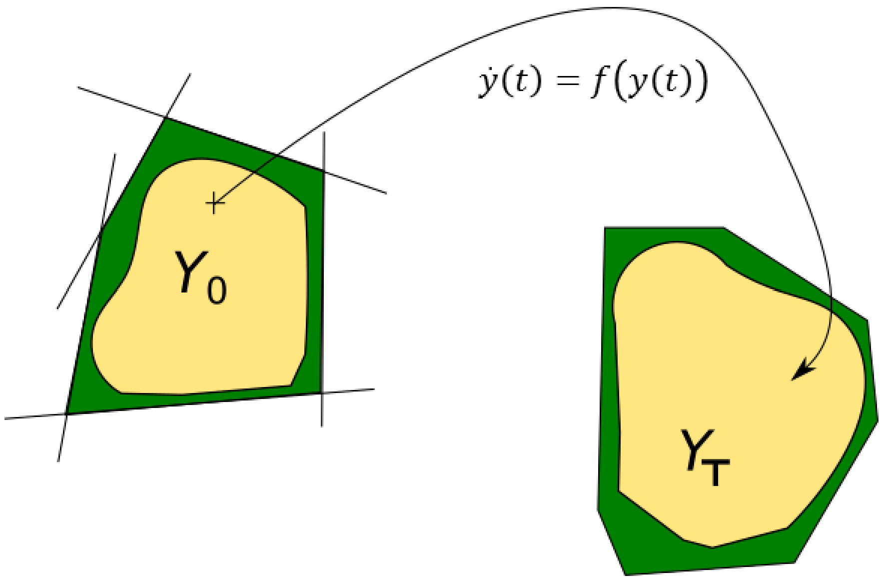
Outline

- Initial Value Problems
- Set Computation Techniques
- Zonotopic Based Validated Runge-Kutta
- Nonlinear ODE Reachability of Polytopes
- Conclusion and Future Work

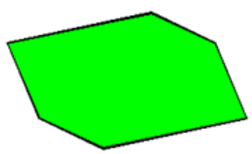
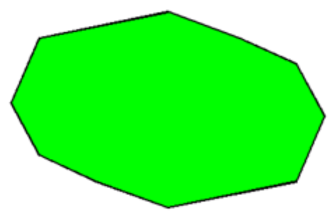
Initial Value Problems

- The initial value problem for a dynamic system $\dot{y}(t) = f\big(y(t)\big),\ \ y(0) = y_0$:
  - If the initial value $y(0)$ is a set Y_0, i.e. $\dot{y}(t) = f\big(y(t)\big), y(0) \in Y_0$
  - Then the solution $y(\mathrm{T})$ at a given time instant T will also be a set $Y_\mathrm{T} = \{y(\tau), y(0) \in Y_0\}$
- How to compute the solution set Y_T is a reachability problem involving **set computation**



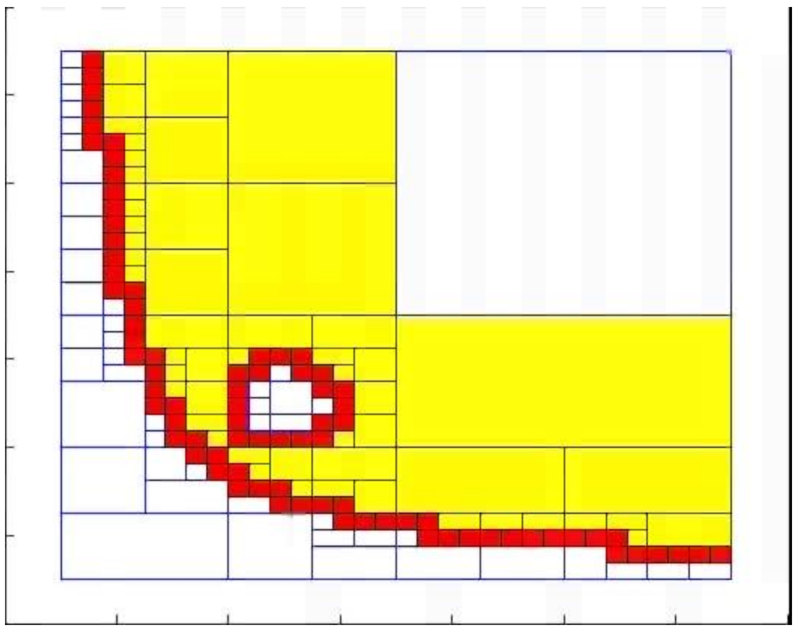
Set Computation Techniques

- For a nonlinear system $X_k = f(X_{k-1})$ where X_{k-1} is a set, currently there are two main set computation techniques:

$X_k = f(X_{k-1})$	**The input set X_{k-1}**	**The output set X_k**	**Methods**
Interval set computation	An interval set: 	Another interval set: 	Using interval arithmetic
Zonotopic set computation	A zonotopic set: 	Another zonotopic set: 	Using Kuhn's method or affine arithmetic

Set Computation Techniques – Interval Arithmetic

- The solution from interval arithmetic can be greatly overestimated, which is called the wrapping effect:
  - For example: $x \in [3,5],\ f(x) = x(10-x) = [3,5] * [5,7] \in [15,35]\ (\mathbf{[21,25]})$
- The correlation between two computing quantities has not been tracked in interval arithmetic, which contributes to the wrapping effect.



Bisection and selection for **finer** set computation

$x \in [3,5] = [3,4] \cup [4,5]$

$x(10-x) = [3,4] * [6,7] \cup [4,5] * [5,6]$
$= [18,28] \cup [20,30]$
$= [18,30]\ (\mathbf{[21,25]})$

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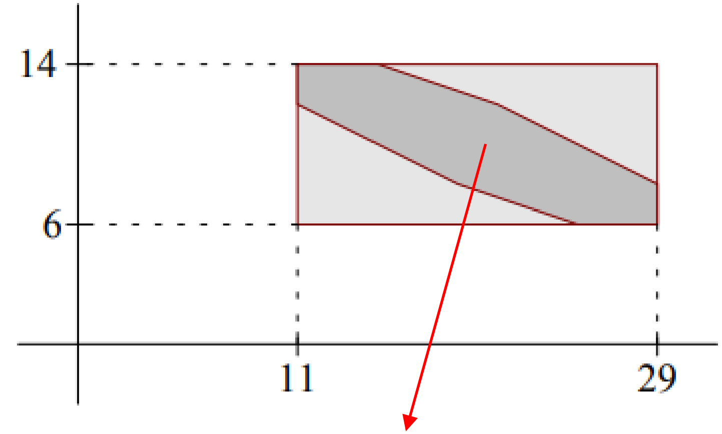
Set Computation Techniques – Affine Arithmetic

- Affine arithmetic is an extension of interval arithmetic by representing an interval by a first-order polynomial, i.e. $[x] = x_0 + x_1 * [-1,1] + x_2 * [-1,1] + \cdots + x_n * [-1,1]$
  - For example: $x \in [3,5],\ x(10-x) = (4 + [-1,1]) * (6 - [-1,1]) \in [21,26]$ $([21,25])$
- Affine arithmetic is also closely related to zonotope geometry as its high-order representation is a zonotope.

$$[x] = 20 - 4\varepsilon_1 + 2\varepsilon_3 + 3\varepsilon_4,\ \varepsilon_{1,2,3,4} = [-1,1]$$

$$[y] = 10 - 2\varepsilon_1 + \varepsilon_2 - \varepsilon_4$$



This is a zonotope!

Set Computation Techniques – Zonotopic Set Computation

- A zonotope is a centrally symmetric convex polytope with an advantageous mathematical representation, i.e. $[\mathbf{x}] = \mathbf{p} + \mathbf{h}_1 * [-1,1] + \cdots + \mathbf{h}_m * [-1,1]$

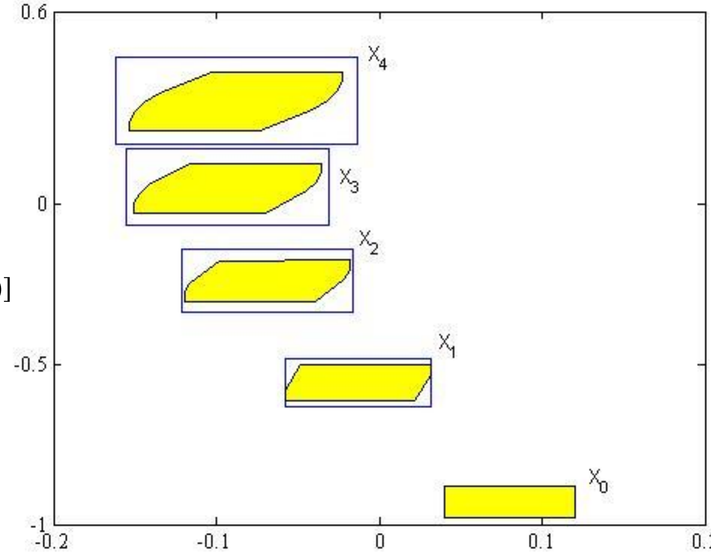
$$[x] = 20 - 4\varepsilon_1 + 2\varepsilon_3 + 3\varepsilon_4,\ \ \varepsilon_{1,2,3,4} = [-1,1]$$

$$[y] = 10 - 2\varepsilon_1 + \varepsilon_2 - \varepsilon_4$$

$$[\mathbf{x}] = \begin{bmatrix}20\\10\end{bmatrix} + \begin{bmatrix}-4\\-2\end{bmatrix} * [-1,1] + \begin{bmatrix}0\\1\end{bmatrix} * [-1,1] + \begin{bmatrix}2\\0\end{bmatrix} * [-1,1] + \begin{bmatrix}3\\-1\end{bmatrix} * [-1,1]$$

- Using Kuhn's method or affine arithmetic, the evolution of a zonotopic set for a nonlinear system can be computed and the wrapping effect can be reduced compared to interval arithmetic.

$$\begin{cases} x_1(k+1) = x_1(k) + 0.1x_2(k) \\ x_2(k+1) = x_2(k) + 0.1[x_1^2(k) + x_2^2(k) + u(k)] \end{cases}$$



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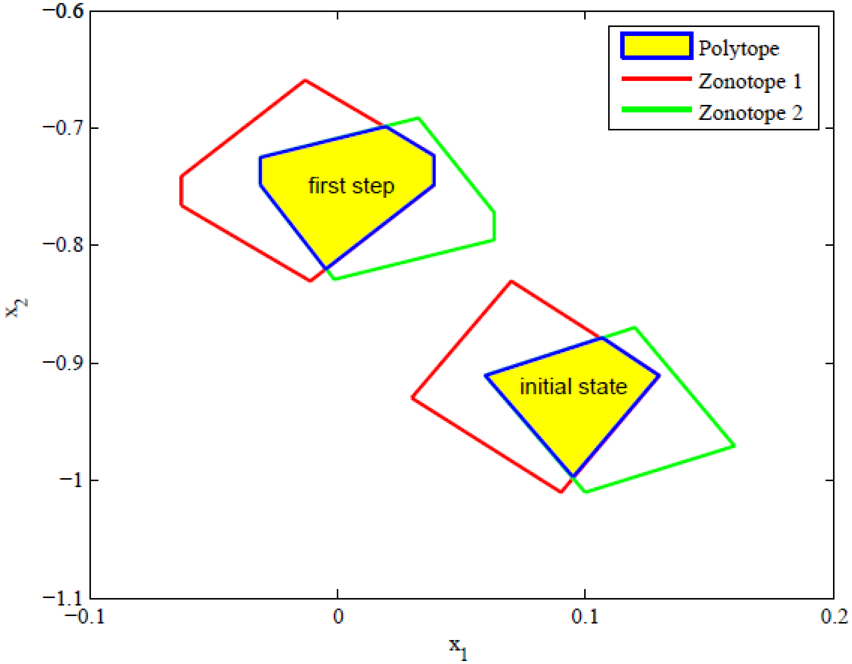
Set Computation Techniques – Polytopic Set Computation

- Polytopic set computation cannot be computed directly for a nonlinear system $X_k = f(X_{k-1})$ where X_{k-1} is a polytope. However, representing the polytope exactly by the intersection of zonotopes, polytopic set computation can be implemented indirectly via zonotopic set computation for a nonlinear system:



$$f(P) = f(z_1 \cap z_2) \subseteq f(z_1) \cap f(z_2)$$

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Zonotopic based Validated Runge-Kutta

- The initial value problem for a dynamic system $\dot{y}(t) = f\big(y(t)\big),\ \ y(0) = y_0$:
  - The initial value $y(0)$ is a set Y_0, where Y_0 is a zonotope
  - A Runge-Kutta scheme $y_{l+1} = y_l + h\sum_{i=1}^{s} b_i\, k_i, k_i = f\big(t_l + c_i h, y_l + h\sum_{j=1}^{s} a_{ij}\, k_j\big)$
  - Coefficients a_{ij}, b_i and c_i are given by Butcher tableau

Runge-Kutta 4

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b} \end{array}$$

0	0	0	0	0
1/2	1/2	0	0	0
1/2	0	1/2	0	0
1	0	0	1	0
	1/6	1/3	1/3	1/6

- The scheme can be computed iteratively using affine arithmetic for tracking correlations and the local truncation error is to be bounded by the Picard-Lindelof operator (all implemented in Dynibex http://perso.ensta-paristech.fr/~chapoutot/dynibex/).

Zonotopic based Validated Runge-Kutta – Example

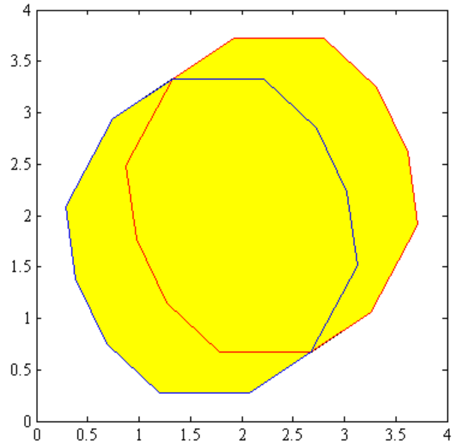
- The Volterra system: $\begin{cases} \dot{y_1} = 2y_1(1-y_2) \\ \dot{y_2} = -y_2(1-y_1) \end{cases}, Z_0 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \otimes \begin{bmatrix} 0.04 & 0.02 \\ 0.02 & 0.02 \end{bmatrix} B^2$

- One-step reachable set with $h = 0.00245s$:

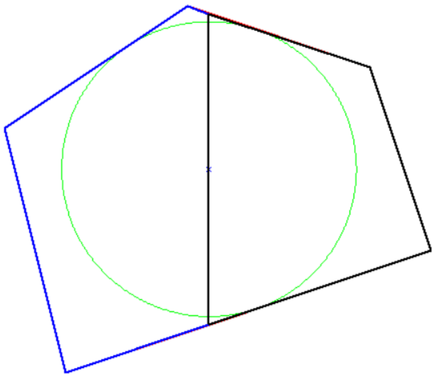
$$y(h) = \begin{bmatrix} 0.9902440 \\ 2.9999640 \end{bmatrix} \otimes \begin{bmatrix} 0.0395119 & 0.01970740 \\ 0.0202920 & 0.02014574 \end{bmatrix} B^2 + \begin{pmatrix} [-2e-06, 2e06] \\ [-2e-06, 2e06] \end{pmatrix}$$

Nonlinear ODE Reachability of Polytopes – Principle

- The initial value problem for a dynamic system $\dot{y}(t) = f\big(y(t)\big),\ \ y(0) = y_0$:
  - The initial value $y(0)$ is a set Y_0, where Y_0 is a polytope P
- The principle for nonlinear ODE reachability of polytopes is as follows:
  - First, represent the polytope by the intersection of m zonotopes $P = Z_1 \cap \ldots \cap Z_m$;
  - Second, compute the individual reachable set $\mathrm{Reach}(Z_i, h)$ for each zonotope
    - If the integration fails, bisect the zonotope or the original polytope and then to compute the union of reachable sets
  - Third, compute the intersection of all reachable sets $\mathrm{Reach}(P) = \mathrm{Reach}(Z_1) \cap \ldots \cap \mathrm{Reach}(Z_m)$



Bisect a zonotope



Bisect a polytope

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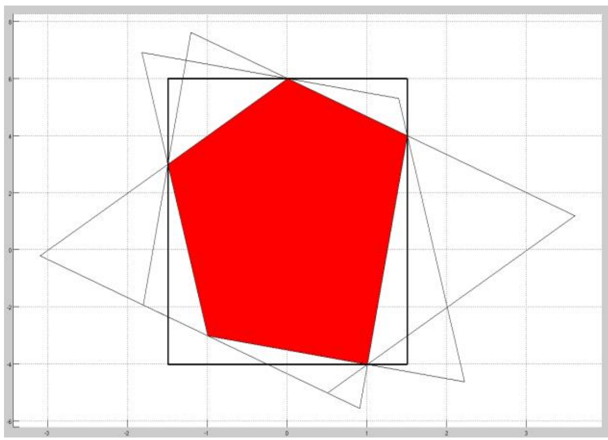
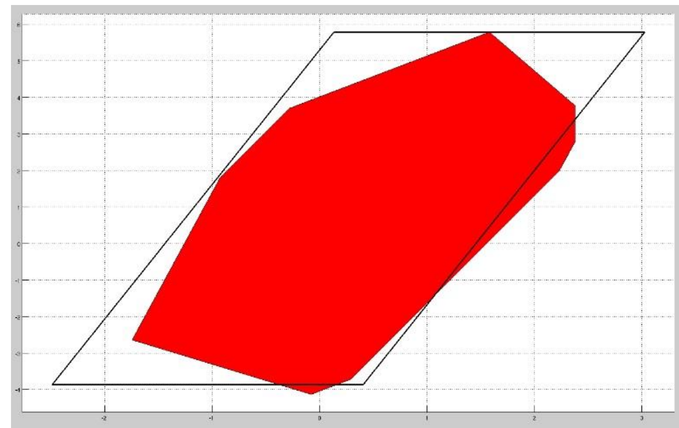
Nonlinear ODE Reachability of Polytopes – Example 1

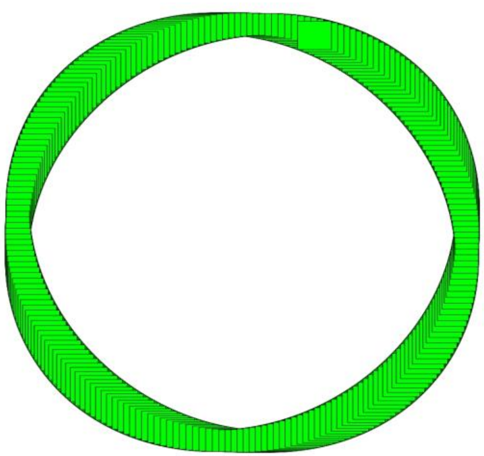
- The circle problem $\begin{cases} \dot{y_1} = 2y_1(1 - y_2) \\ \dot{y_2} = -y_2(1 - y_1) \end{cases}$:



The initial set of a polytope



The solution set at $t = 50s$



Initial polytope (IP)	21.25
Initial hull (IH)	30
Polytope (P) from IP	22.5046
Zonotope (Z) from IH	27.9348
Intersection of P and Z	21.8109

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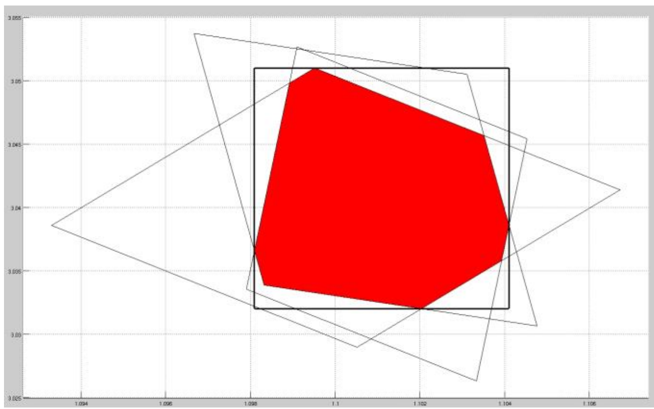
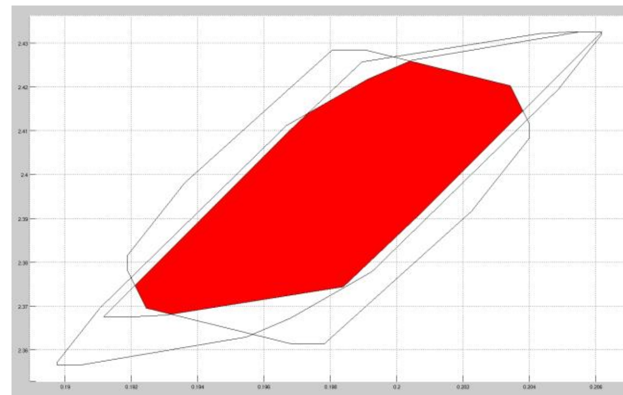
Nonlinear ODE Reachability of Polytopes – Example 2

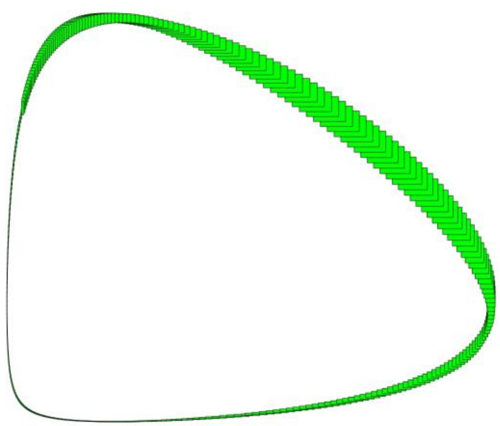
- The Volterra system $\begin{cases} \dot{y_1} = 2y_1(1 - y_2) \\ \dot{y_2} = -y_2(1 - y_1) \end{cases}$:



The initial set of a polytope



The solution set at $t = 6s$



Initial polytope (IP)	8.2505e-05
Initial hull (IH)	1.1400e-04
Polytope (P) from IP	3.2273e-04
Zonotope (Z) from IH	5.8018e-04
Intersection of P and Z	3.0337e-04

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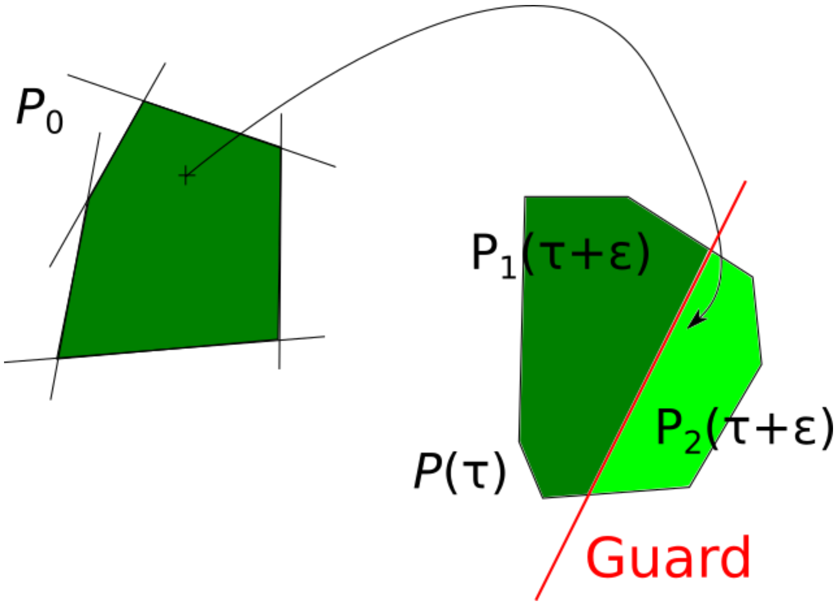
Conclusion and Future Work

- Built on zonotopic based validated Runge-Kutta method and polytope geometry
- Capable of computing reachability of nonlinear ODEs with a polytope as the initial value
- Have good results in terms of volumes for second-order systems

Conclusion and Future Work

- Apply to more examples of higher dimension and benchmark problems
- Reachability problems for hybrid systems
- Constrained reachability problems



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Thank you!
Any questions?

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