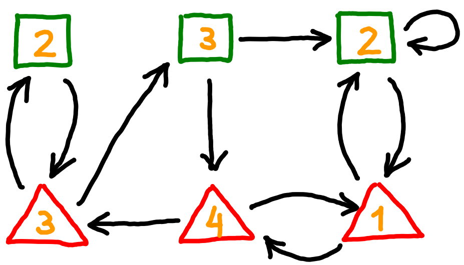
NEW ALGORITHMS FOR PARITY GAMES



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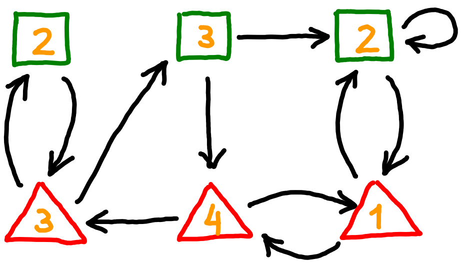
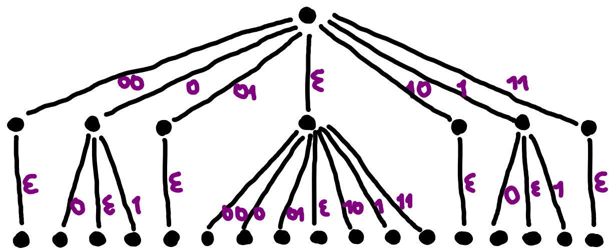
COLLABORATORS: W. CZERWIŃSKI, L. DAVIAUD, N. FIJALKOW, **R. LAZIĆ**, P. PARYS

UNIVERSAL TREES AND QUASI-POLYNOMIAL ALGORITHMS FOR SOLVING PARITY GAMES





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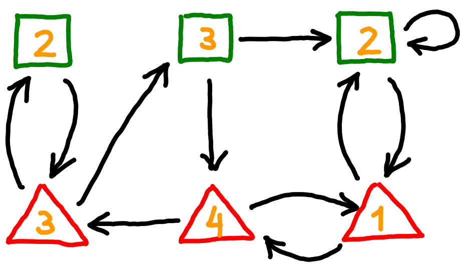
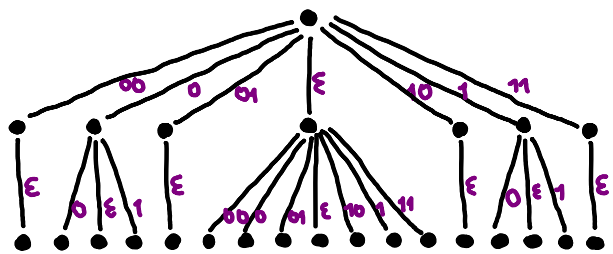
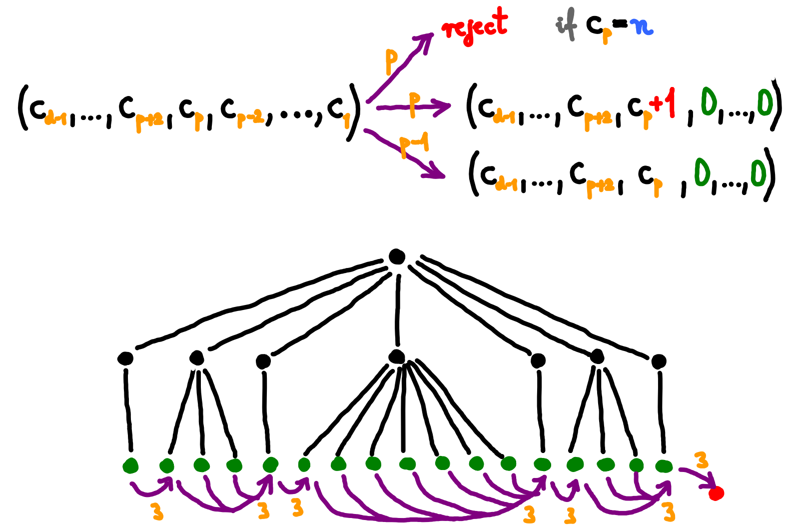
Parity games, universal trees, and separating automata







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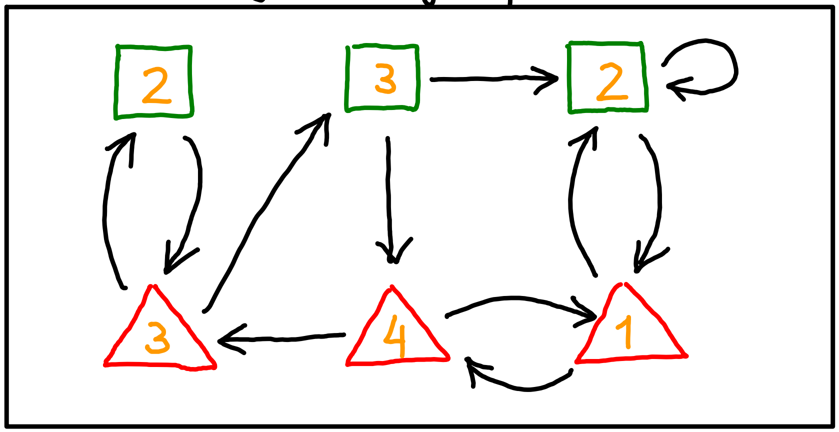
Collaborators: W. Czerwiński, **L. Daviaud**, N. Fijalkow, **R. Lazić**, P. Parys

PARITY GAMES

$n = |V|$

$m = |E|$

A game graph



$$G = (V = V_{Even} \uplus V_{Odd}, E, \pi)$$

$$\pi : V \to \{1, 2, 3, 4, 5, \dots, d\}$$

PARITY GAMES

$n = |V|$

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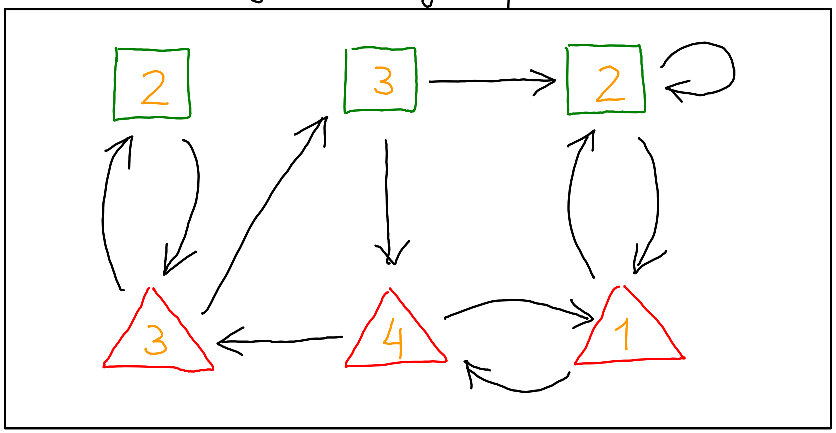
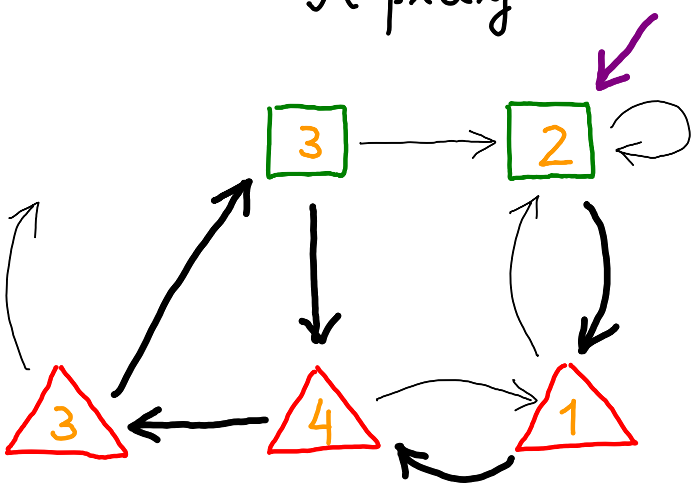


$$G = (V = V_{Even} \uplus V_{Odd}, E, \pi)$$

$$\pi : V \to \{1, 2, 3, 4, 5, \dots, d\}$$

A play



Even wins $\langle p_1, p_2, p_3, \dots \rangle$

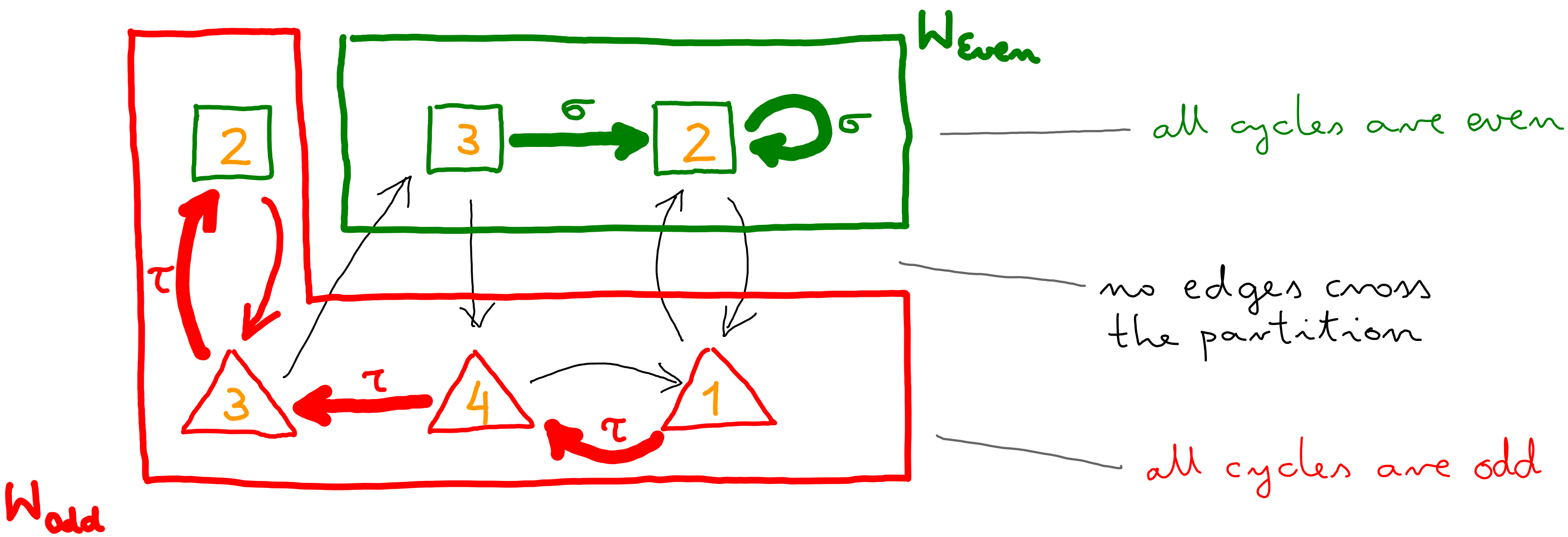
iff

$\left[\limsup_{i \to \infty} p_i\right]$ is even

SHORT WITNESSES FROM POSITIONAL DETERMINACY

THEOREM [Emerson, Jutla 1991; Mostowski 1991; re-proved since 1960's]

Parity games are **positionally determined**



COROLLARY [Emerson, Jutla, Sistla 1993]

Deciding the winner in parity games is $\boxed{\text{in } NP \cap co\text{-}NP}$

ALGORITHMS FOR SOLVING PARITY GAMES

- $\boxed{n^{d+O(1)}}$ [McNaughton 1993 ; Zielonka 1998]
- $\boxed{n^{\frac{d}{2}+O(1)}}$ [Browne, Clarke, Jha, Long, Marrero 1994; Seidl 1996; J. 2000]
- $\boxed{n^{d+O(1)}}$ strategy iteration [Vöge, J. 2000]

  $\boxed{2^{\Omega(n)}}$ [Friedmann 2009]

  - policy iteration for MDPs [Fearnley 2010]
  - randomized simplex [Hansen, Friedmann, Zwick 2011]
- $\boxed{n^{O(\sqrt{n})}}$ [Björklund, Sandberg, Vorobyov 2003; J., Paterson, Zwick 2006]
- $\boxed{n^{\frac{d}{3}+O(1)}}$ [Schewe 2007]

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- $n^{\frac{d}{3}+O(1)}$ [Schewe 2007]
- $n^{\lg d+O(1)}$ [Calude, Jain, Khoussainov, Li, Stephan 2017; J., Lazić 2017; Lehtinen 2018]

GAMES FOR SYNTHESIS

specification

φ

open system

G

Synthesize

a **controller** for G
that satisfies φ

GAMES FOR SYNTHESIS

open system

specification

φ

$$\varphi \mapsto \mathcal{A}_\varphi$$

$$\underbrace{G \times \mathcal{A}_\varphi}_{\text{synchronized product}}$$

deterministic parity automaton

parity game

Synthesize

a controller for G that satisfies φ

a **winning strategy** in $G \times \mathcal{A}_\varphi$

AUTOMATA FOR GAMES

[Bernet, Janin, Walukiewicz 2002; Bojańczyk, Czerwiński 2018]

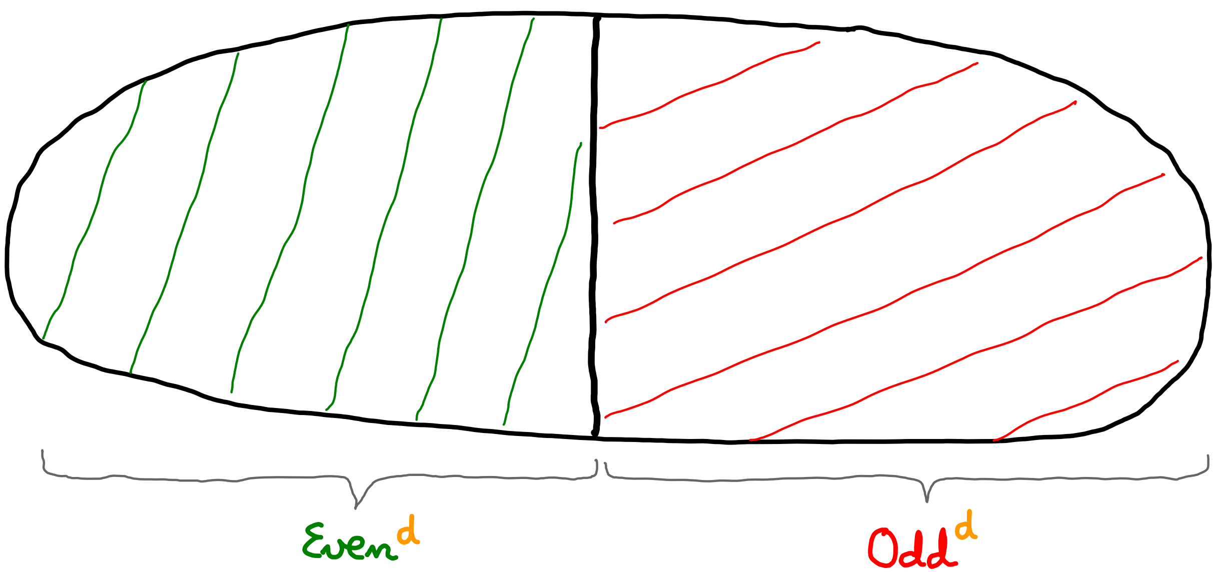
parity game — G × A — deterministic **reachability**/**safety** automaton that "approximates" the parity condition

Question: What "approximation" properties should A have, so that games G and G×A have the **same winner**?

LANGUAGES OF PRIORITY SEQUENCES

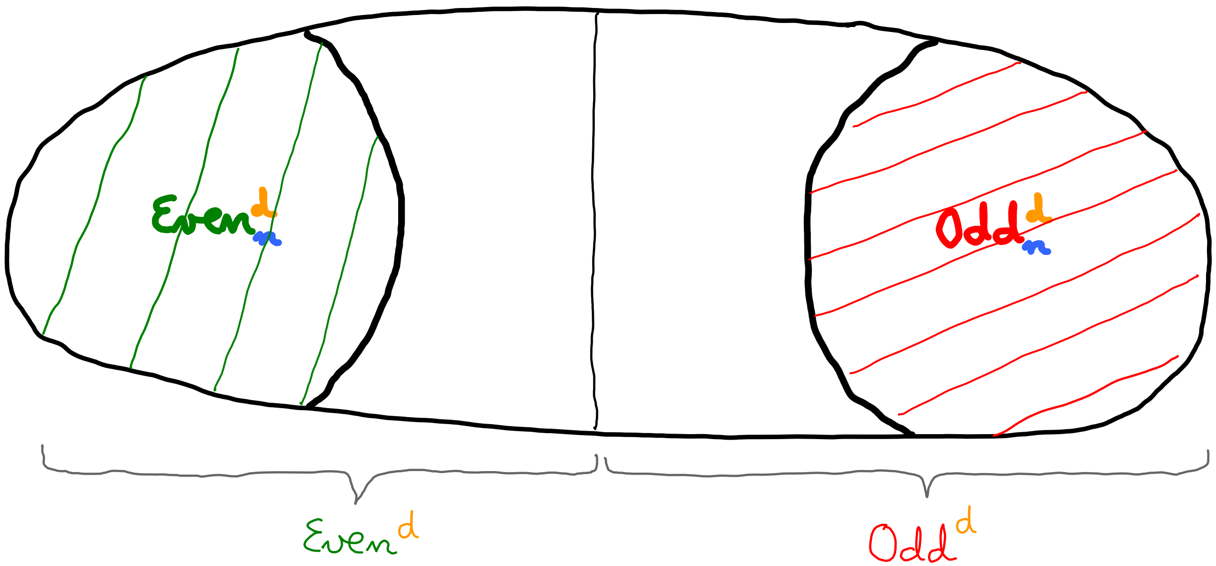
- $\mathrm{Even}^{d} \subseteq \{1, 2, \ldots, d\}^{\omega}$: won by Even

---



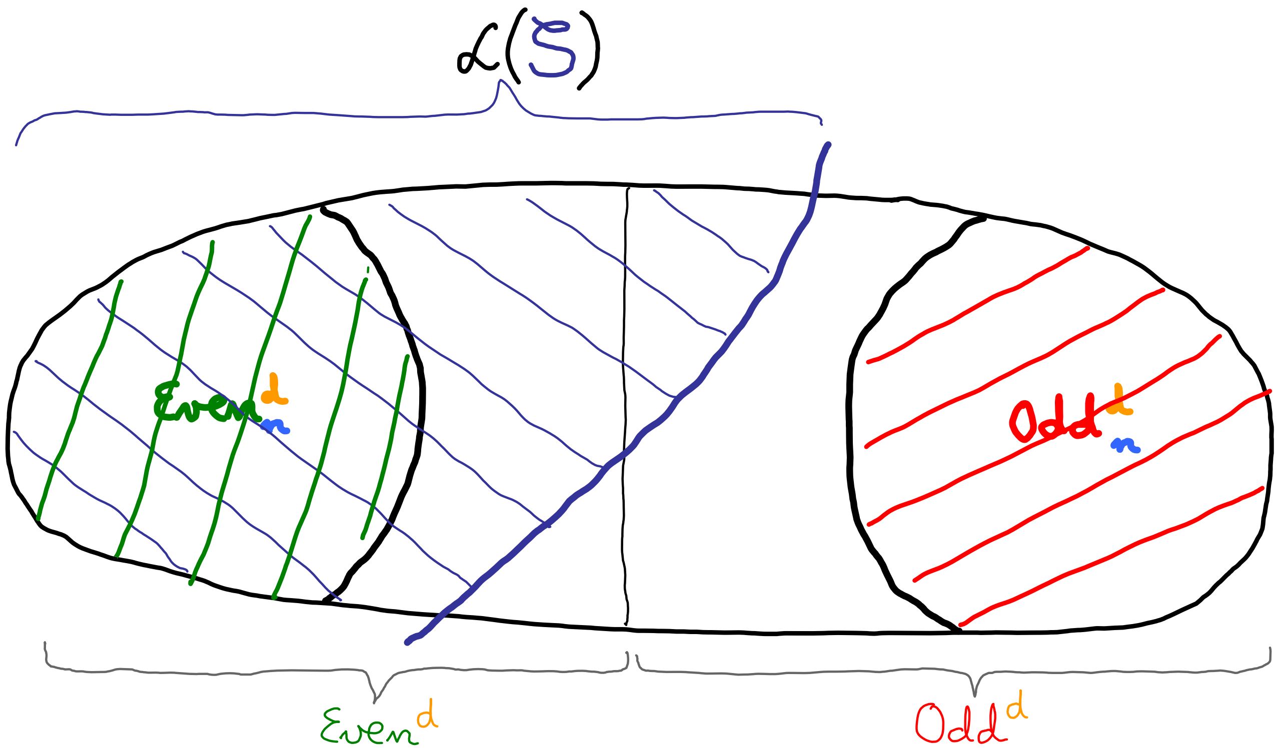
LANGUAGES OF PRIORITY SEQUENCES

- $\mathrm{Even}^d \subseteq \{1,2,\ldots,d\}^\omega$: won by Even
- $\mathrm{Even}^d_n \subseteq \{1,2,\ldots,d\}^\omega$: arising from a game graph (with ≤n vertices and ≤d priorities) in which all cycles are even



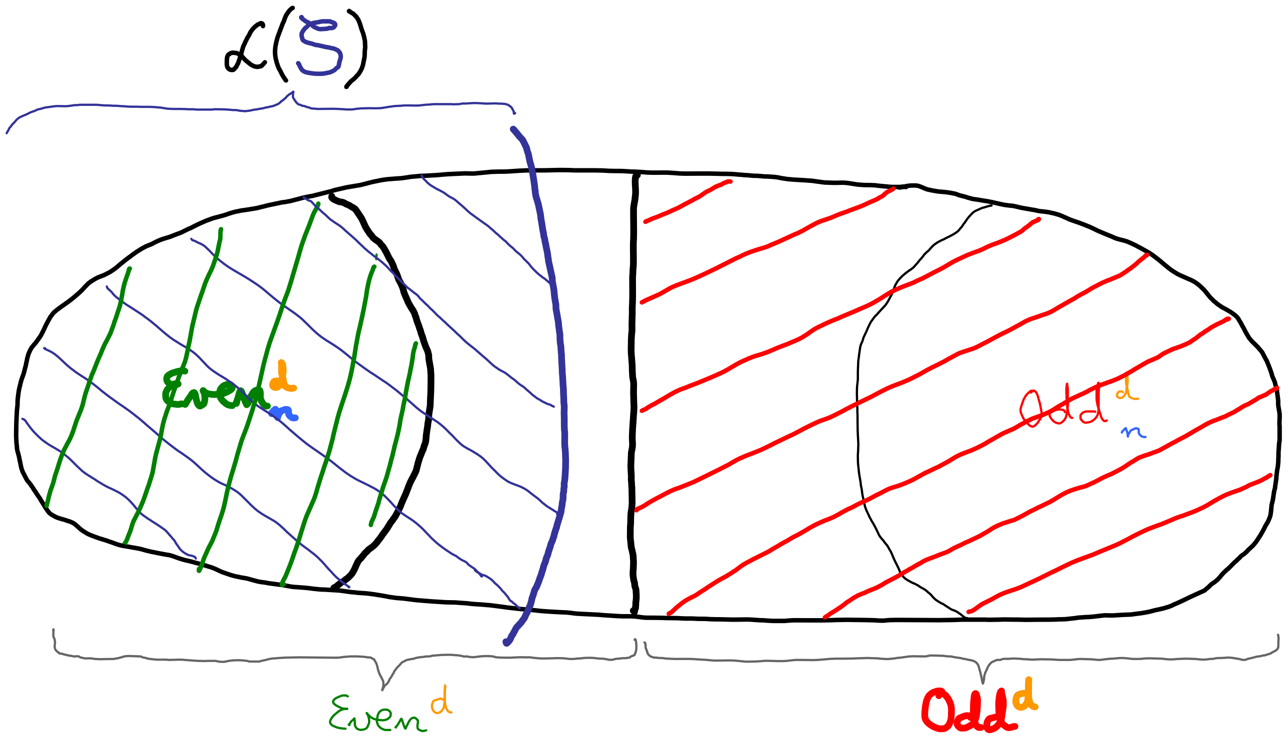
SEPARATING (SAFETY) AUTOMATA

DEFINITION A finite (safety) automaton $\mathcal{S}$ is an (n,d)-separator if $\mathcal{L}(\mathcal{S})$ separates Even_n^d from Odd_n^d



SEPARATING (SAFETY) AUTOMATA

DEFINITION A finite (safety) automaton $\mathcal{S}$ is an strong (n,d)-separator if $\mathcal{L}(\mathcal{S})$ separates Even^d_n from Odd^d



THE SEPARATION APPROACH

The synchronized product $G \times S$ is a safety game.

parity game

safety automaton

FACT If S is an (n,d)-separator
and the parity game G has $\leq n$ vertices and $\leq d$ priorities
then games G and $G \times S$ have the same winners

parity game

safety game

MULTI-COUNTER SEPARATOR OF SIZE $n^{d/2}$

[Bernet, Janin, Walukiewicz 2003]

- **States**: multi-counters $(c_{d-1}, c_{d-3}, \dots, c_3, c_1)$ s.t. $0 \leqslant c_p \leqslant n$
- **Initial state**: $(0, \dots, 0)$
- **Transitions**:

$$(c_{d-1}, \dots, c_{p+2}, c_p, c_{p-2}, \dots, c_1) \xrightarrow{p} \text{reject} \quad \text{if } c_p = n$$

$$(c_{d-1}, \dots, c_{p+2}, c_p, c_{p-2}, \dots, c_1) \xrightarrow{p} (c_{d-1}, \dots, c_{p+2}, c_p + 1, 0, \dots, 0)$$

$$(c_{d-1}, \dots, c_{p+2}, c_p, c_{p-2}, \dots, c_1) \xrightarrow{p-1} (c_{d-1}, \dots, c_{p+2}, c_p, 0, \dots, 0)$$

A QUASI-POLYNOMIAL SEPARATOR

THEOREM [Calude et al. 2017; Bojańczyk, Czerwiński 2018]
There is a strong (n,d)-separator of size $n^{\lg d + O(1)}$

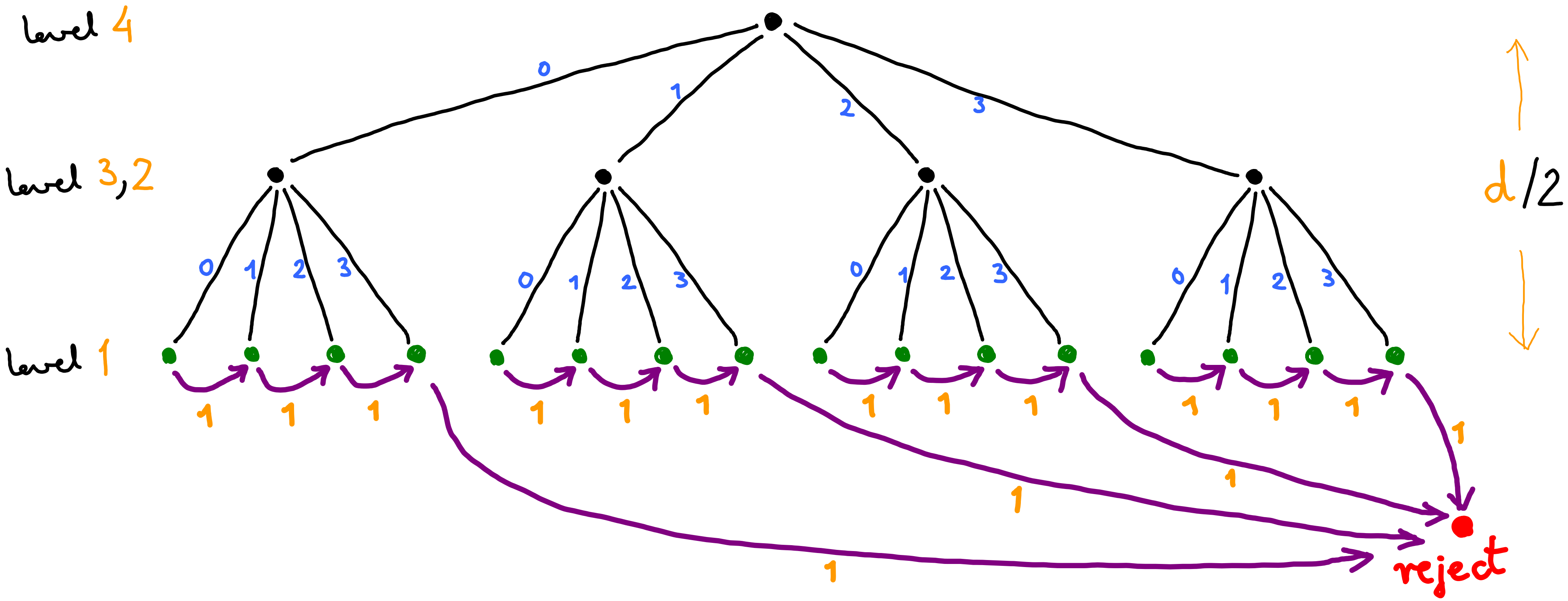
- **States**: "play statistics" $(p_{\lg n}, p_{\lg n - 1}, \ldots, p_1, p_0) \in \{0,1,2,\ldots,d\}^{\lg n+1}$
- **Transitions**: see [Calude et al. 2017] or [Bojańczyk, Czerwiński 2018]

$\lg d \cdot \lg n$-space **TM**

automaton of size $n^{\lg d + O(1)}$

MULTI-COUNTERS AS AN ORDERED TREE

$\mathcal{M}_{3,4}$ — complete 4-ary tree of height 2



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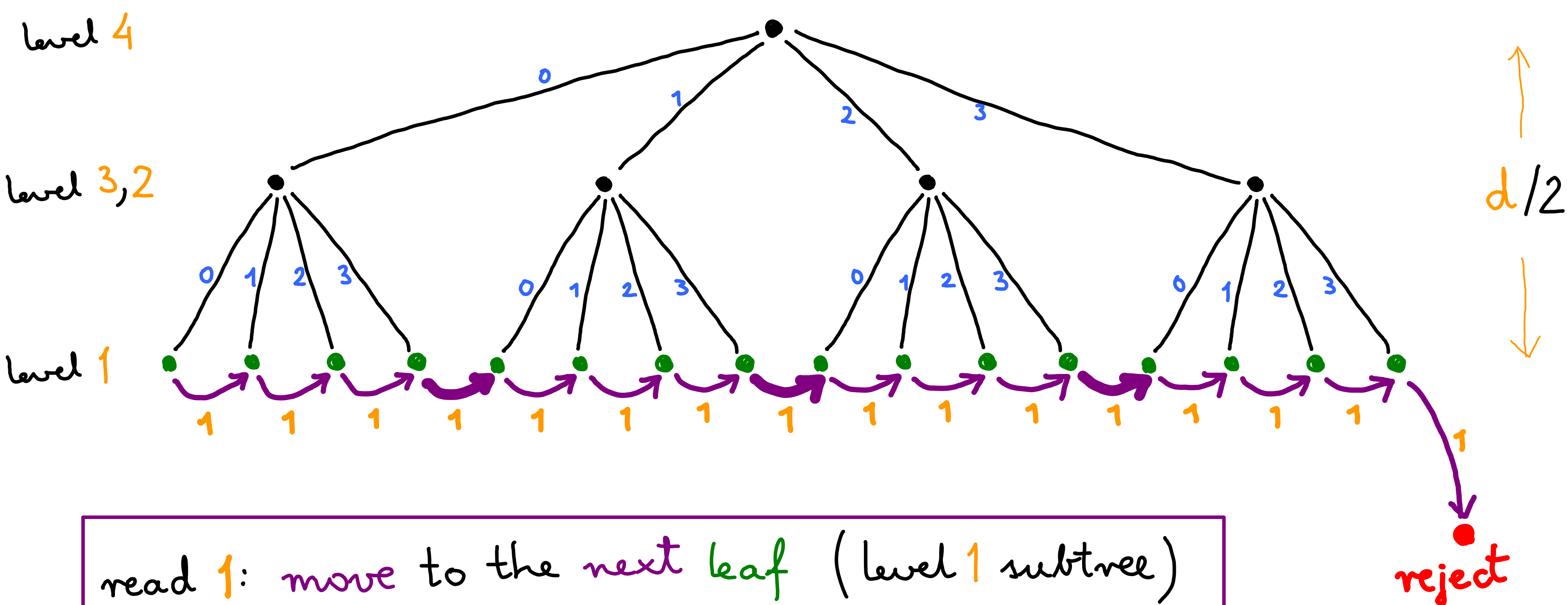

read 1: move to the next leaf (level 1 subtree)

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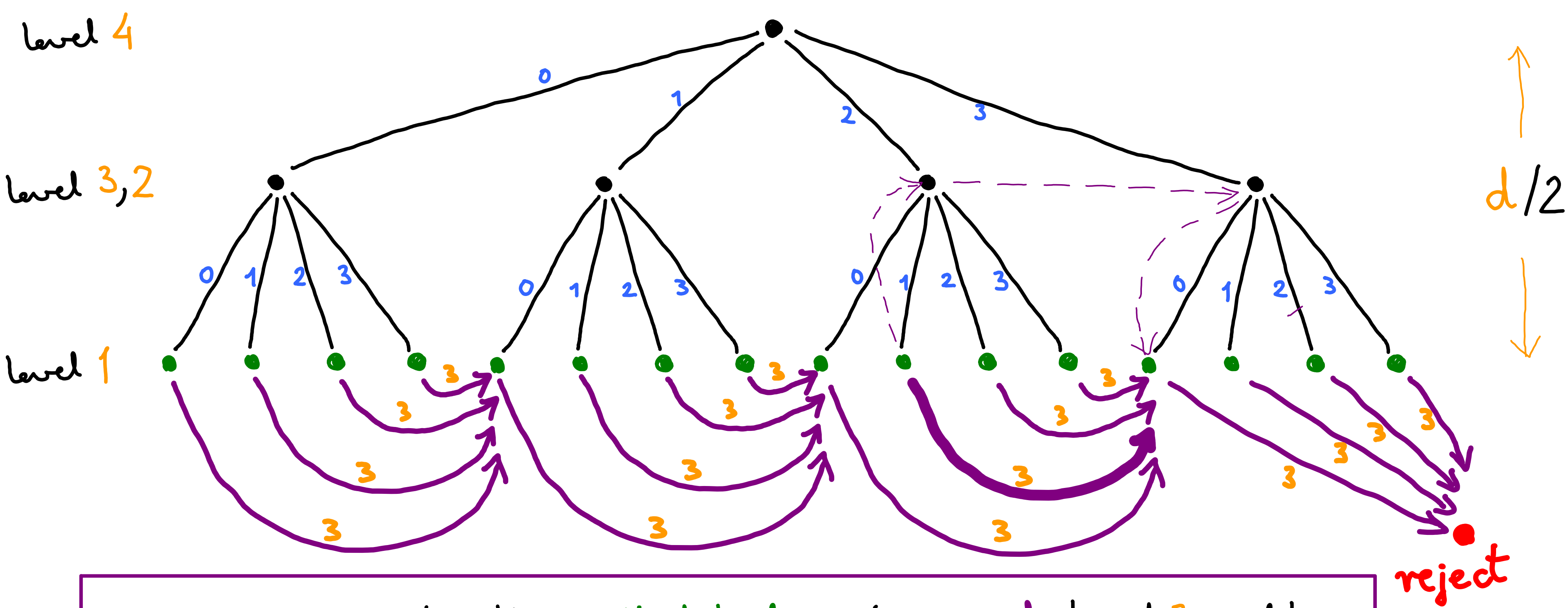
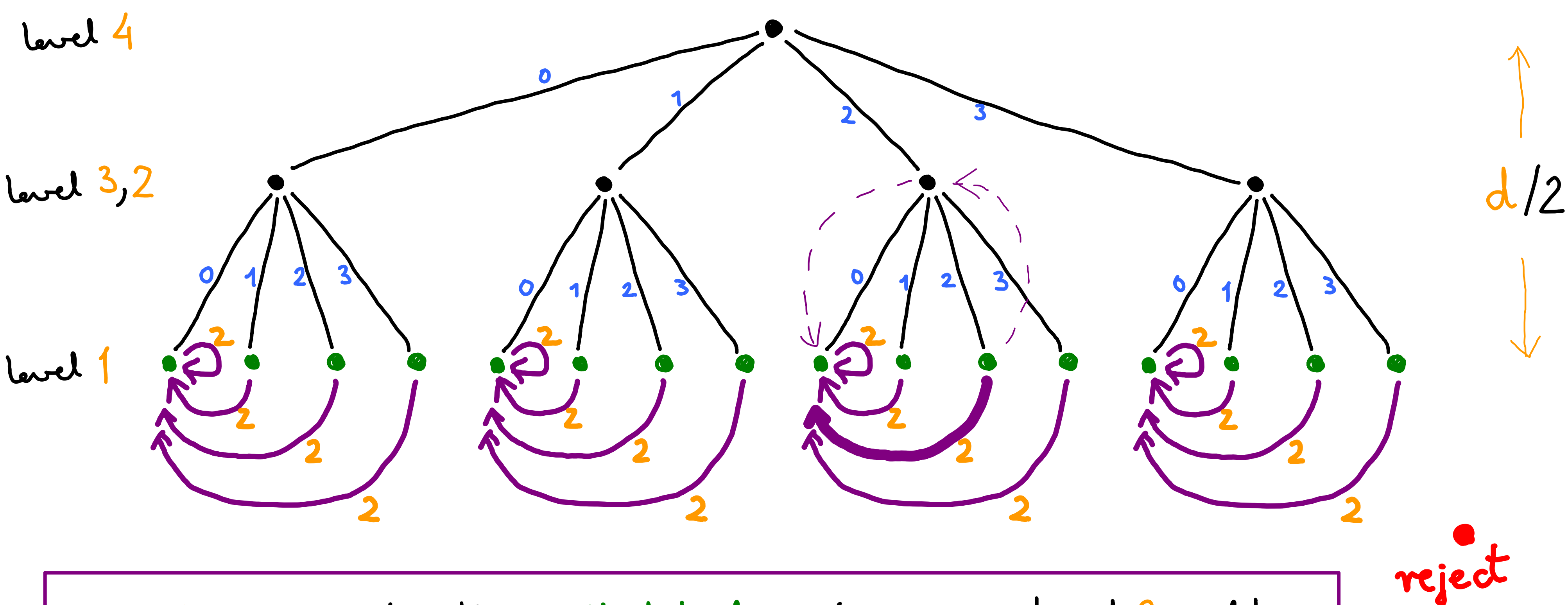


read 3: move to the smallest leaf in the next level 3 subtree

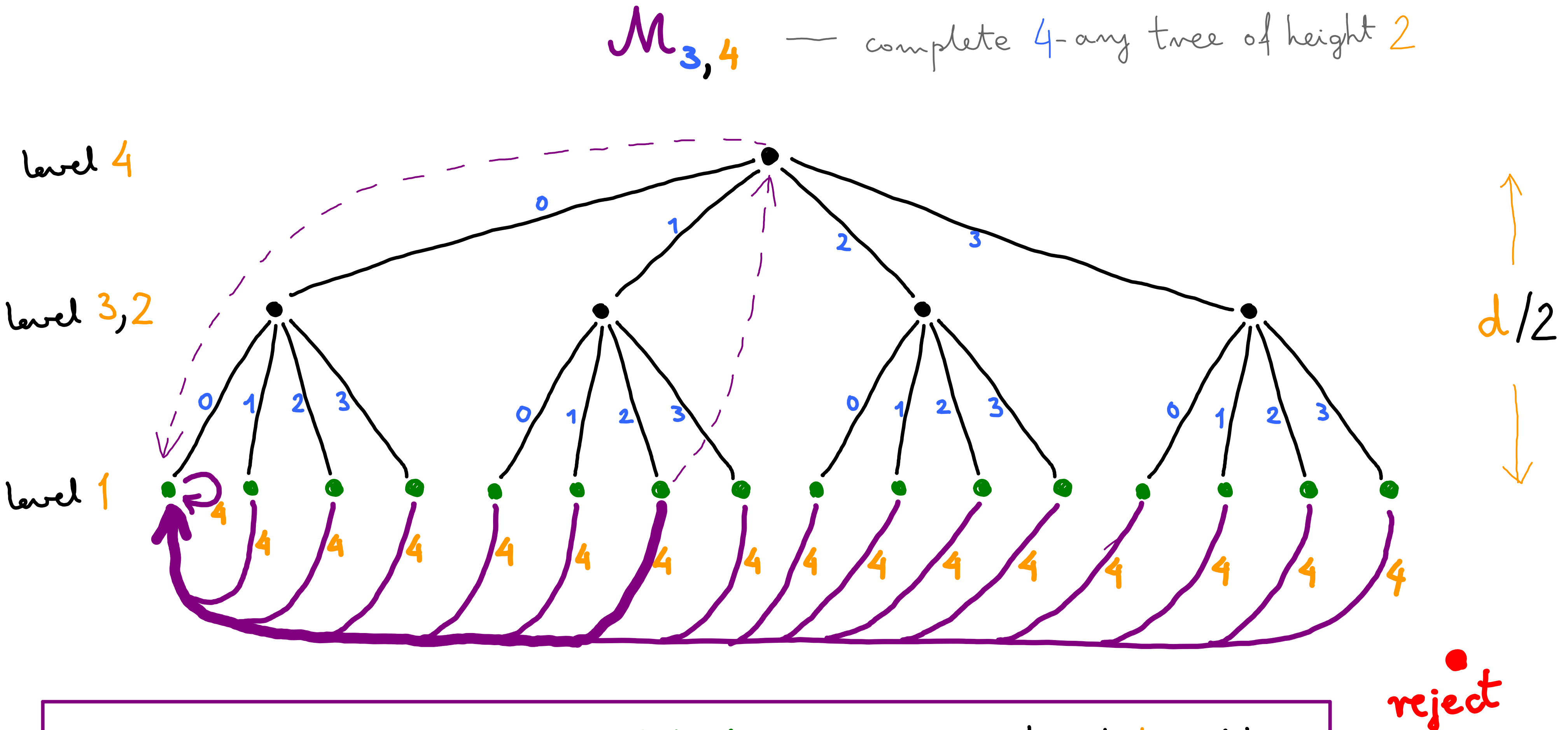
MULTI-COUNTERS AS AN ORDERED TREE

$\mathcal{M}_{3,4}$ — complete 4-ary tree of height 2



read 2: move to the smallest leaf in the same level 2 subtree

MULTI-COUNTERS AS AN ORDERED TREE



read 4: move to the smallest leaf in the same level 4 subtree

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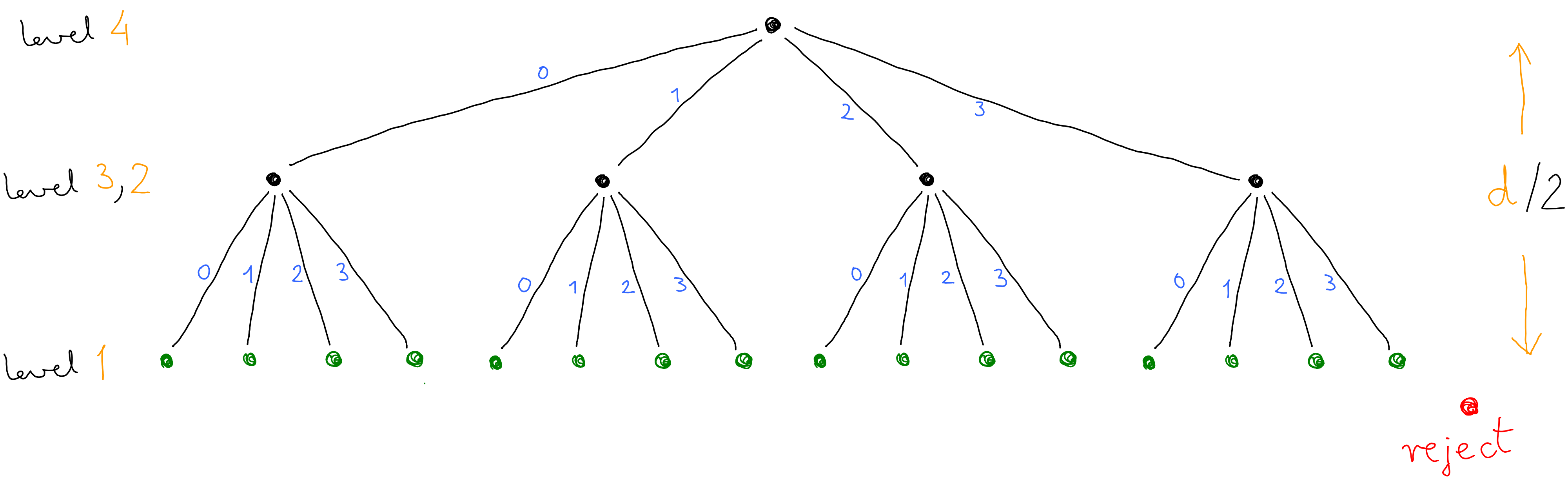


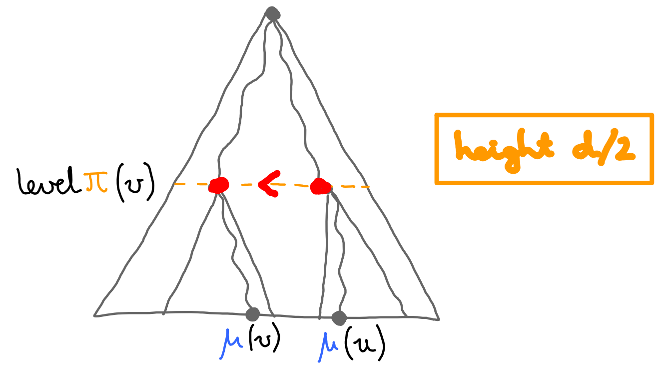
Question: Do we really need the complete $(n+1)$-ary tree of height $\frac{d}{2}$?

TREE WITNESSES

DEFINITION $\mu : V \to \mathcal{T}$ is a tree witness for graph G if

for every edge $(v,u) \in E$,

- $\boxed{\mu(v)|_{\pi(v)} < \mu(u)|_{\pi(v)}}$ if $\pi(v)$ is odd



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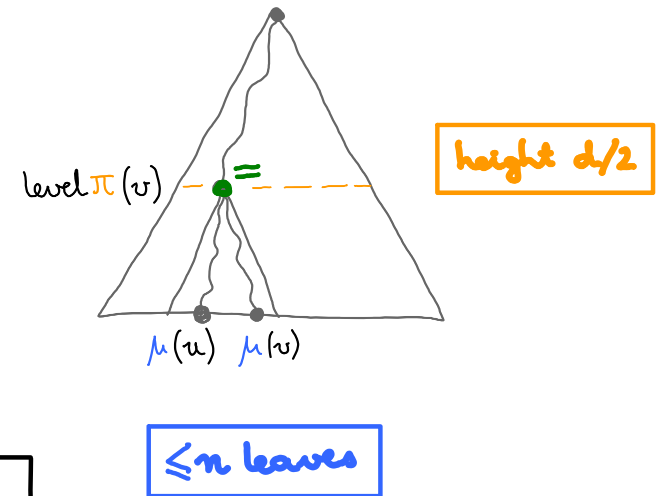

a.k.a. signature assignment, progress measure

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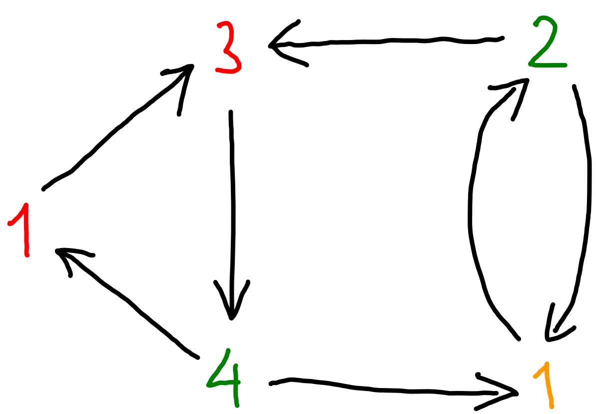

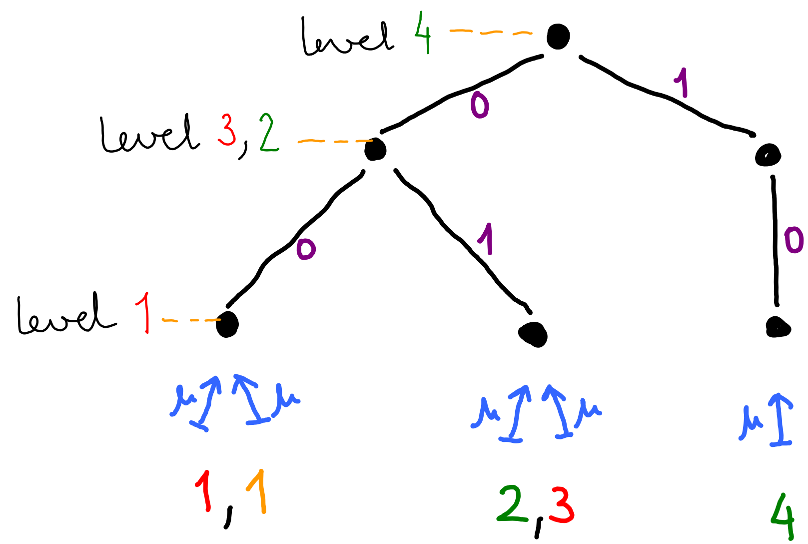

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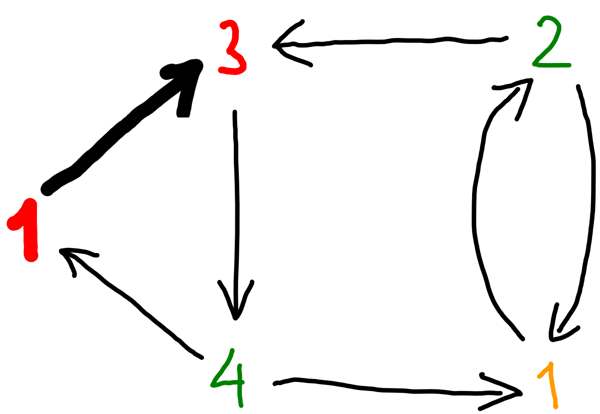

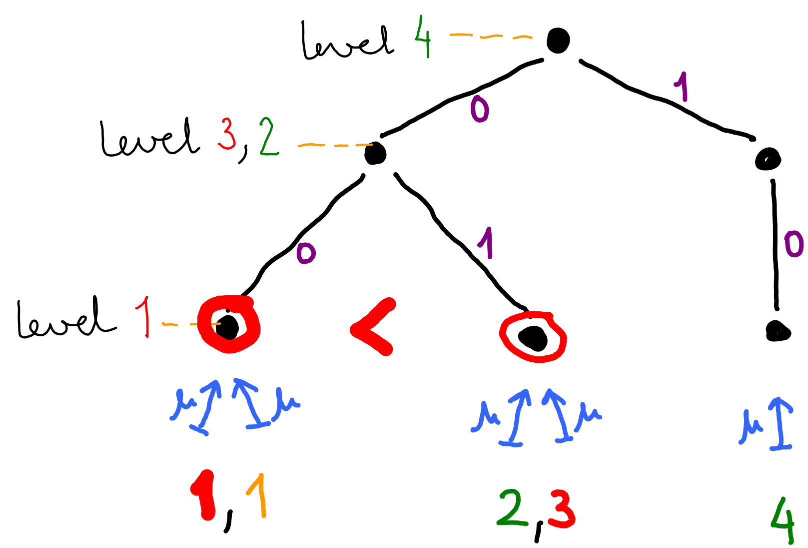

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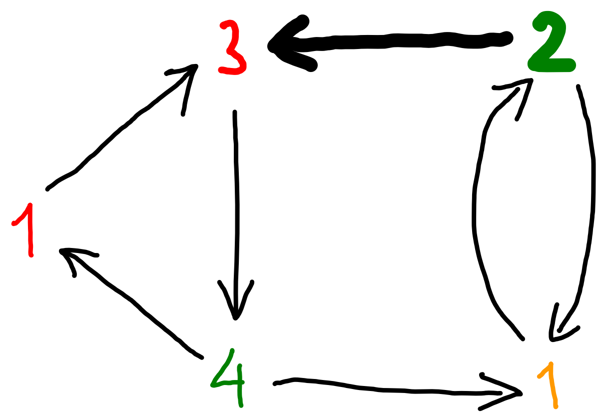
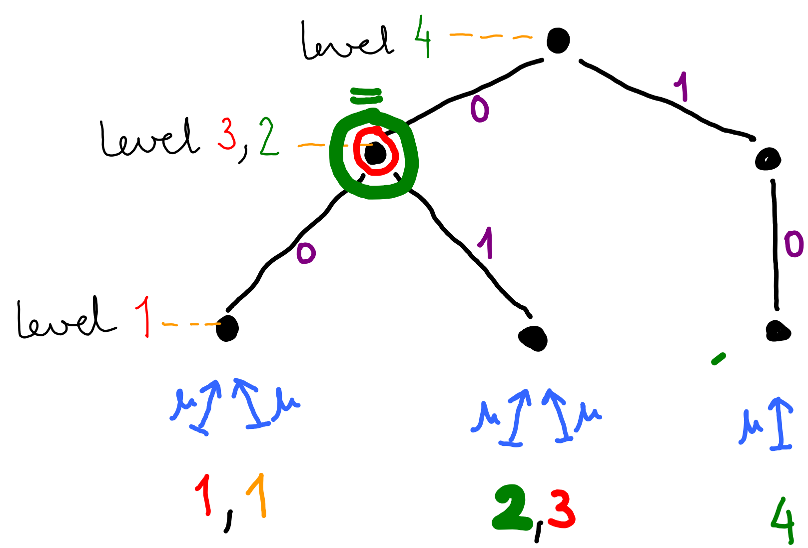
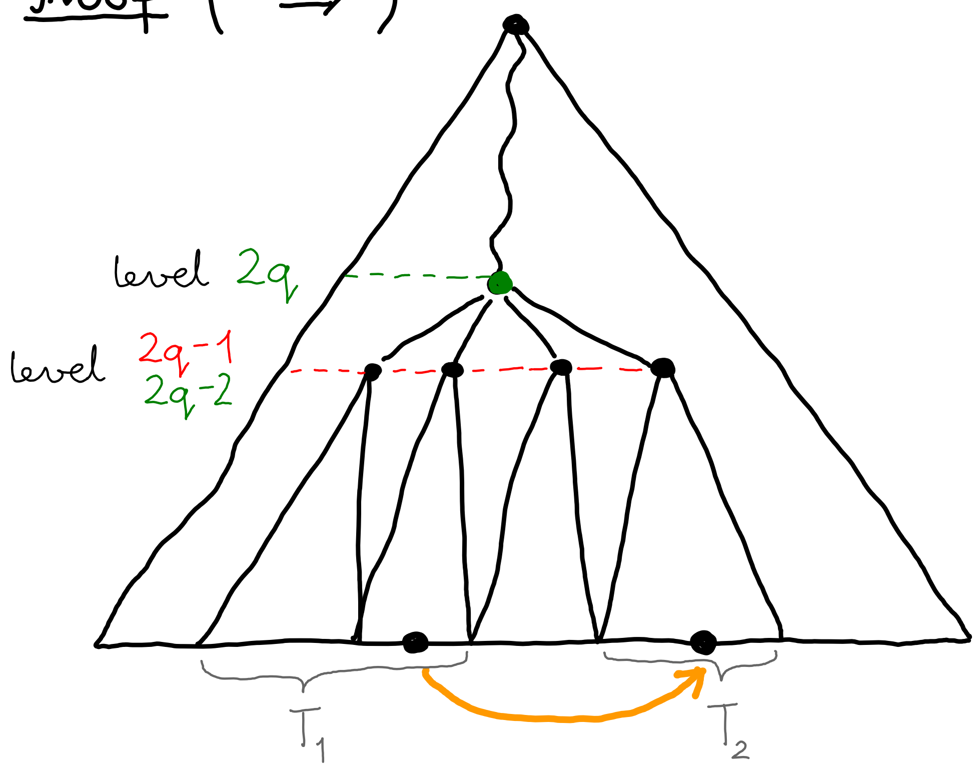




TREE WITNESSES

THEOREM [...; Emerson, Jutla 1991; ...; h/t Gastin 2018]
G has a tree witness iff every cycle in G is even

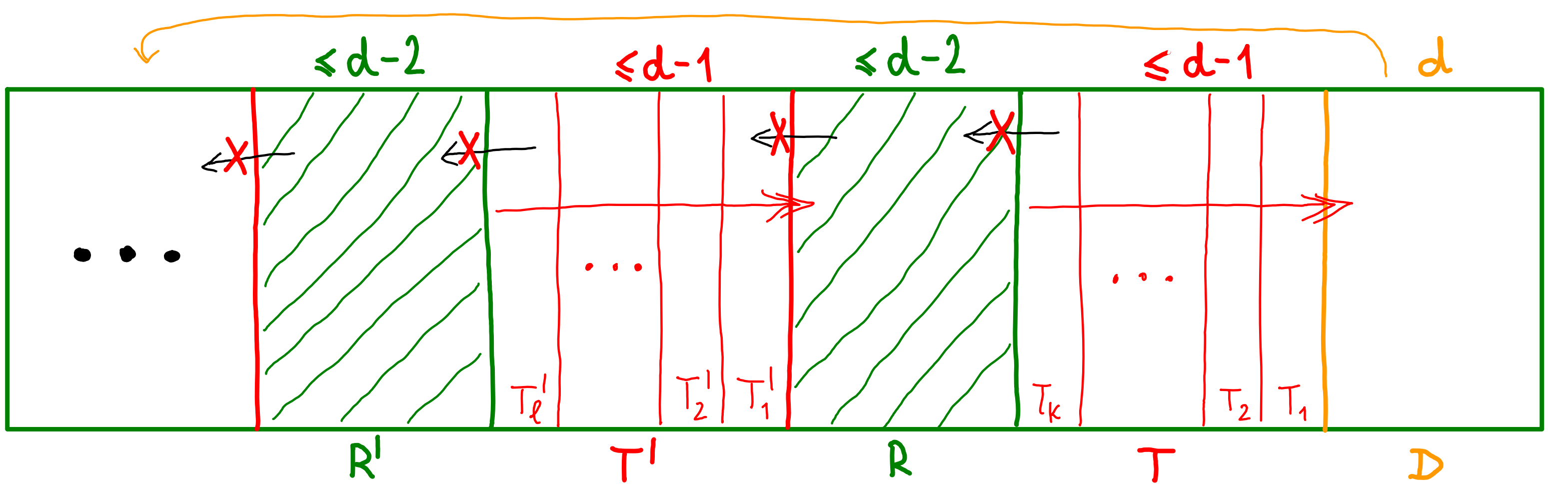
Proof ("$\Rightarrow$")



- all odd priorities on the cycle are $< 2q$
- an even priority $2q$ or higher occurs on the cycle because there is an edge on the cycle from T_2 to T_1

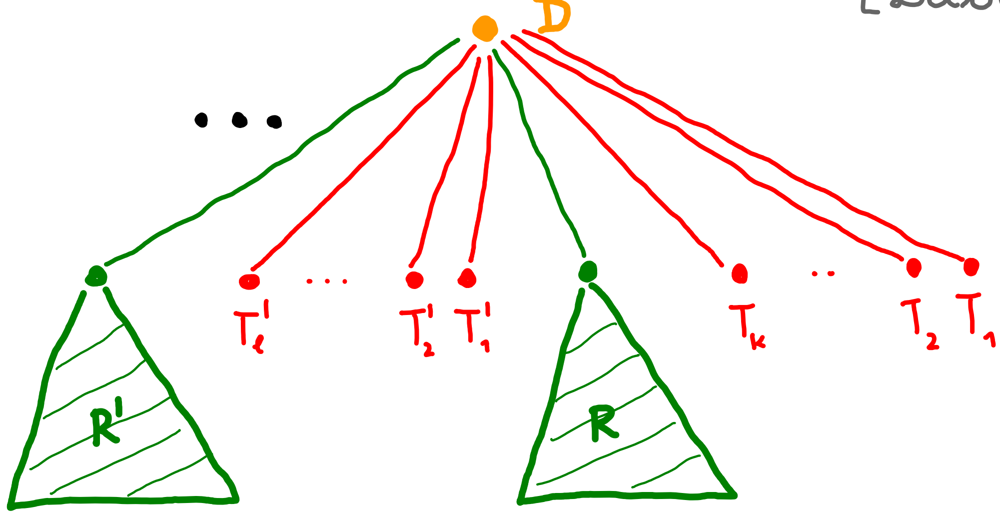
STRATEGY DECOMPOSITIONS

[Daviaud, J., Lazić 2018]



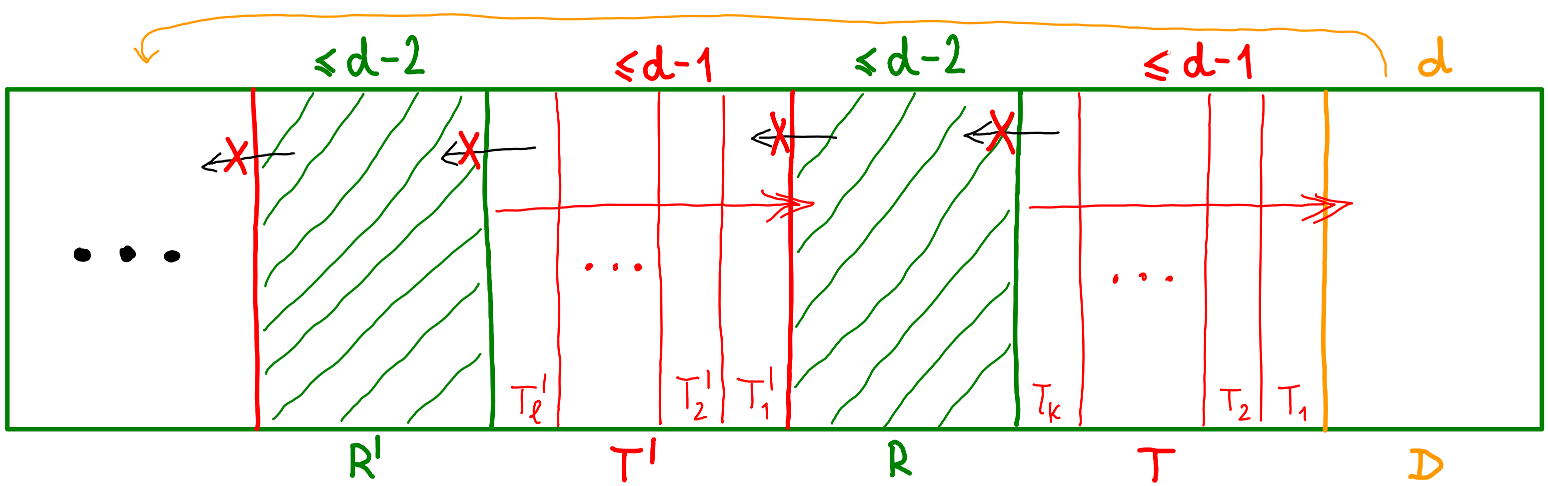
TREE WITNESSES FROM STRATEGY DECOMPOSITIONS

[Daviaud, J., Lazić 2018]



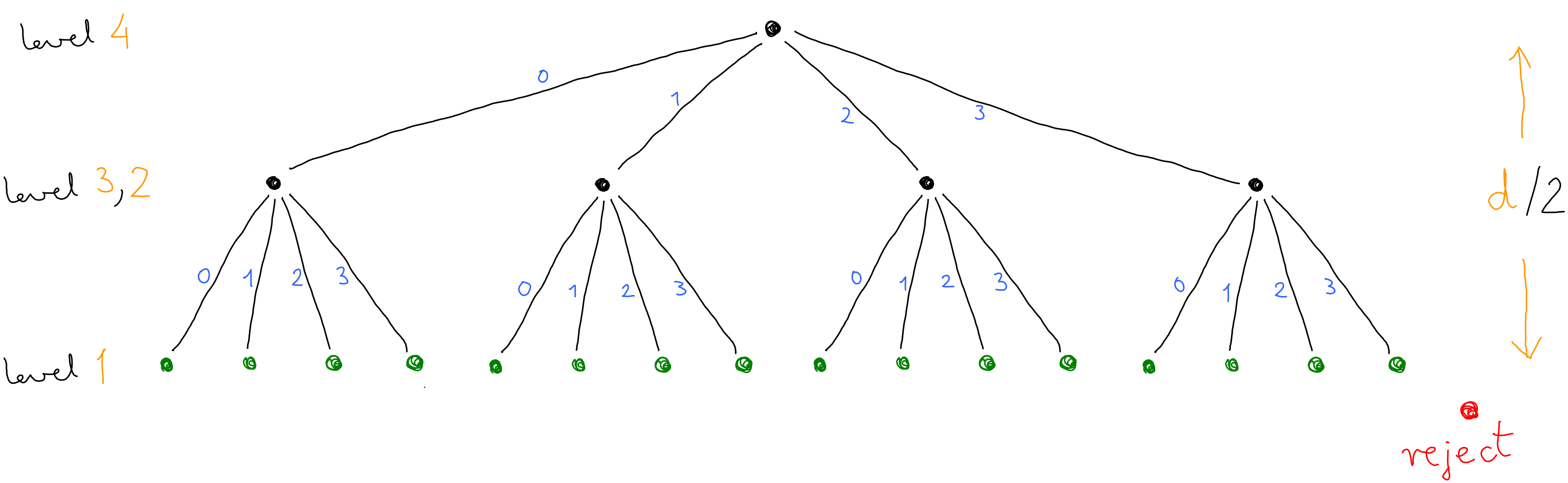
Ordered tree

- height $\leq \frac{d}{2}$
- $\leq n$ leaves



MULTI-COUNTERS AS AN ORDERED TREE

$\mathcal{M}_{3,4}$ — complete 4-ary tree of height 2

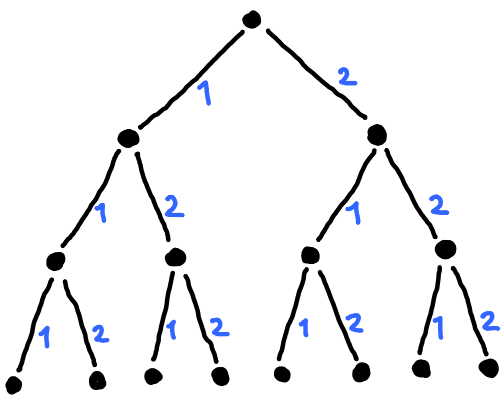


Question: Do we really need the complete $(n+1)$-ary tree of height $\frac{d}{2}$?

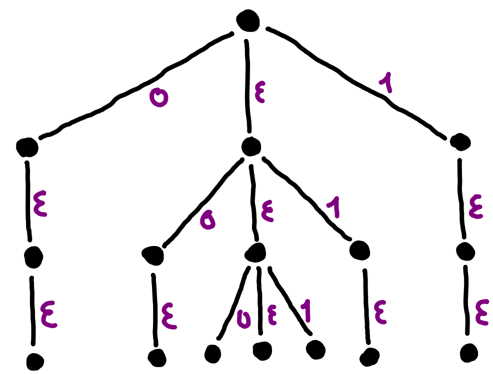
UNIVERSAL ORDERED TREES

DEFINITION $\mathcal{T}$ is an (ℓ, h)-universal ordered tree if every ordered tree of height $\leqslant h$ with $\leqslant \ell$ leaves can be isomorphically embedded into $\mathcal{T}$

$(2,3)$-universal ordered trees



$\boxed{1 < 2}$



$\boxed{0 \prec \varepsilon \prec 1}$

UNIVERSAL ORDERED TREES

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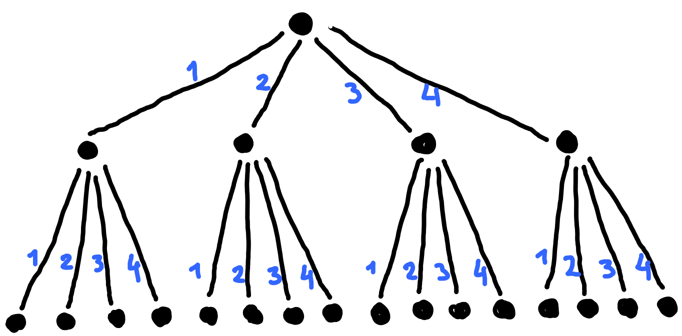
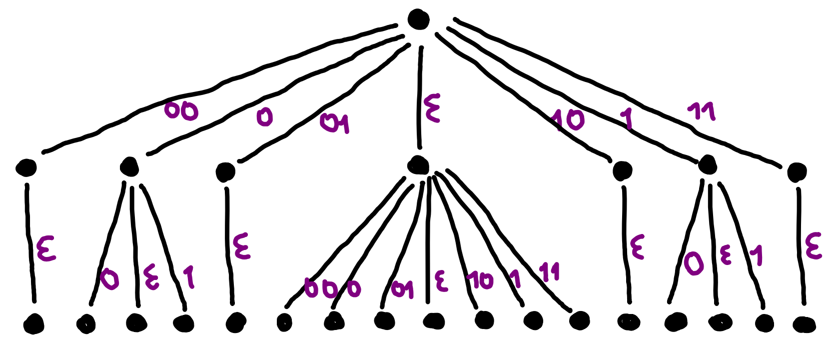
(4,2)-universal ordered trees



$$\boxed{1 < 2 < 3 < 4}$$



$$\boxed{00 \prec 0 \prec 01 \prec \varepsilon \prec 10 \prec 1 \prec 11}$$

UNIVERSAL TREES AND SEPARATING AUTOMATA ARE QUASI-POLYNOMIAL

THEOREM [Czerwiński, Daviaud, Fijalkow, J., Lazić, Parys 2018]

The sizes of smallest universal trees
and of smallest separating automata
are quasi-polynomial

UNIVERSAL TREES ≡ SEPARATING AUTOMATA

THEOREM [Czerwiński, Daviaud, Fijalkow, J., Lazić, Parys 2018]

Leaves of every $(n, \frac{d}{2})$-universal tree are the states of a strong (n, d)-separator

THEOREM [Czerwiński, Daviaud, Fijalkow, J., Lazić, Parys 2018]

States in every strong (n, d)-separator include all the leaves in an $(n, \frac{d}{2})$-universal tree

SMALLEST UNIVERSAL TREES ARE QUASI-POLYNOMIAL

THEOREM [J., Lazić 2017]

There is an $\left(n, \frac{d}{2}\right)$-universal tree

of size $n \cdot \binom{\lg n + \frac{d}{2}}{\lg n} = n^{\lg\left(\frac{d}{\lg n}\right) + O(1)}$

THEOREM [Czerwiński, Daviaud, Fijalkow, J., Lazić, Parys 2018]

Every $\left(n, \frac{d}{2}\right)$-universal tree

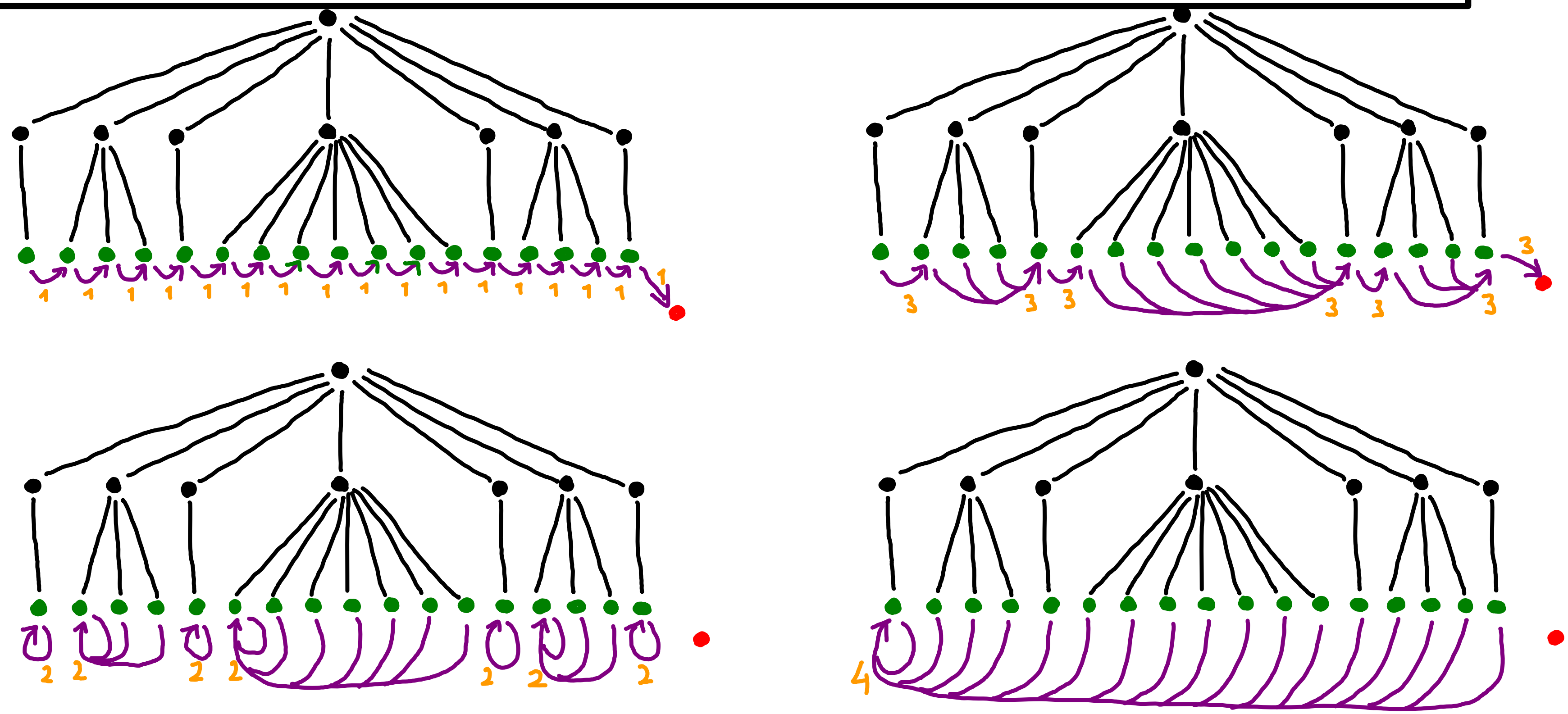
is of size at least $\binom{\lg n + \frac{d}{2} - 2}{\lg n - 1} \geqslant n^{\lg\left(\frac{d}{\lg n}\right) - 2}$

SEPARATING AUTOMATA FROM UNIVERSAL TREES

THEOREM [Czerwiński, Daviaud, Fijalkow, J., Lazić, Parys 2018]

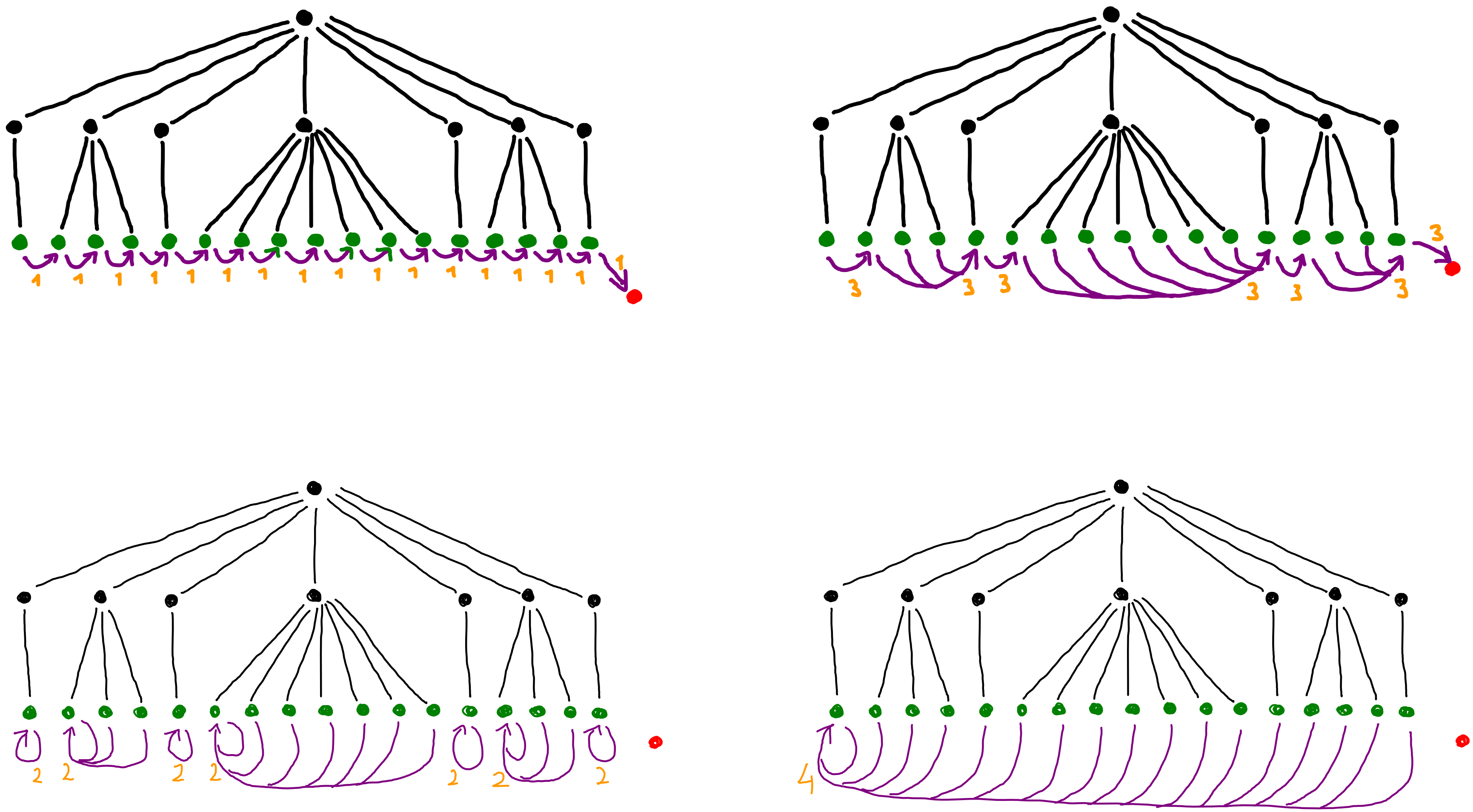
Leaves of every $(n, \frac{d}{2})$-universal tree are the states of a strong (n, d)-separator

$\mathcal{A}_{\mathcal{U}}$:



$\mathcal{A}_u$ REJECTS ALL WORDS IN Odd_d

$\mathcal{A}_u$:



$\mathcal{A}_{\mathcal{U}}$ ACCEPTS ALL WORDS IN $\text{Even}_{n,d}$

- If all cycles in G are even then there is a tree witness

$$\mu : V \longrightarrow T \hookrightarrow \mathcal{U}$$

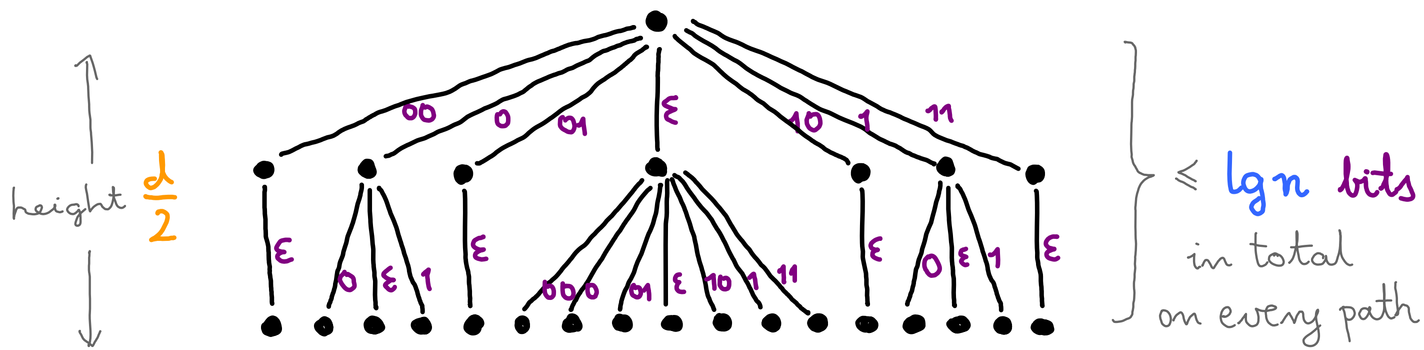
- Let $\mathcal{A}_{\mathcal{U}}$ run on G:

Invariant: $\boxed{\text{when } \mathcal{A}_{\mathcal{U}} \text{ visits vertex } v, \text{ its state is } \leqslant \mu(v)}$

Universal trees: a quasi-polynomial upper bound

Theorem [J., Lazić 2017]

There is an (efficiently navigable) $(n, \frac{d}{2})$-universal ordered tree of size $n \cdot \binom{\lg n + \frac{d}{2}}{\lg n} = n^{\lg\left(\frac{d}{\lg n}\right) + O(1)}$



$(4, 2)$-universal ordered tree

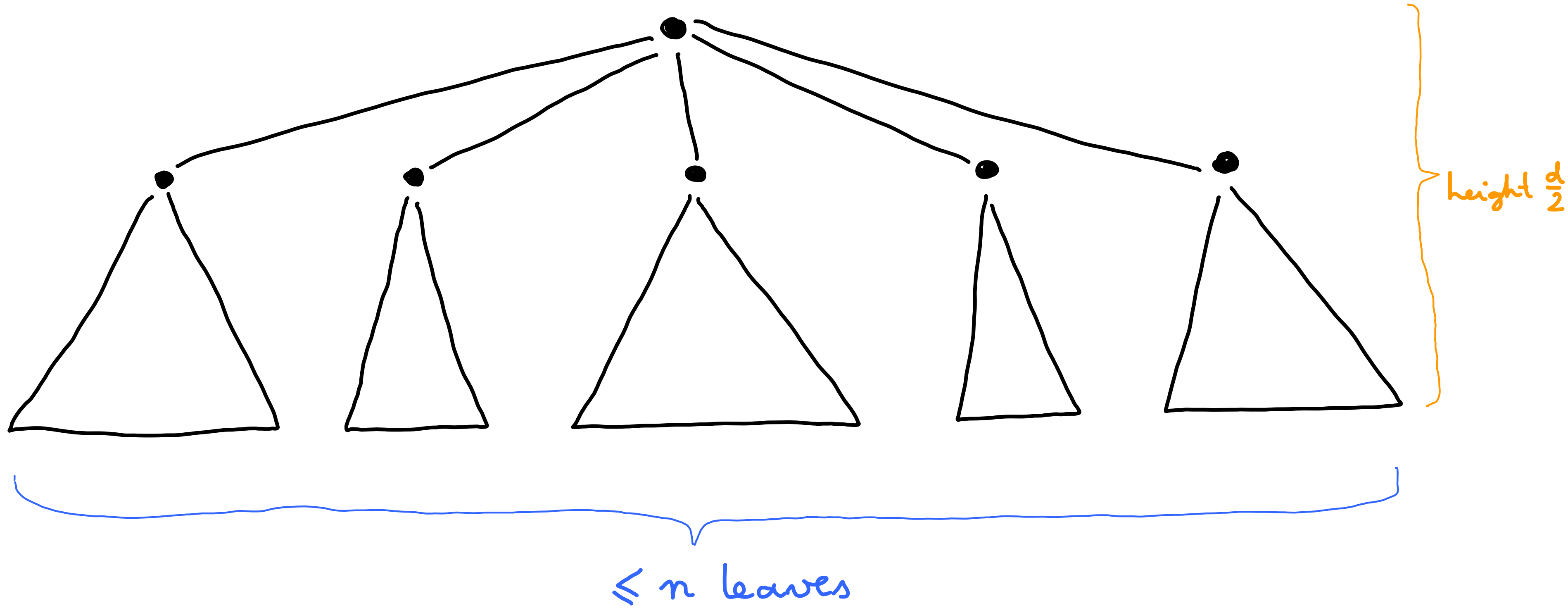
$\boxed{00 < 0 < 01 < \varepsilon < 10 < 1 < 11}$

the linear order on bit strings induced by $0 < \varepsilon < 1$

TREE CODING LEMMA

LEMMA

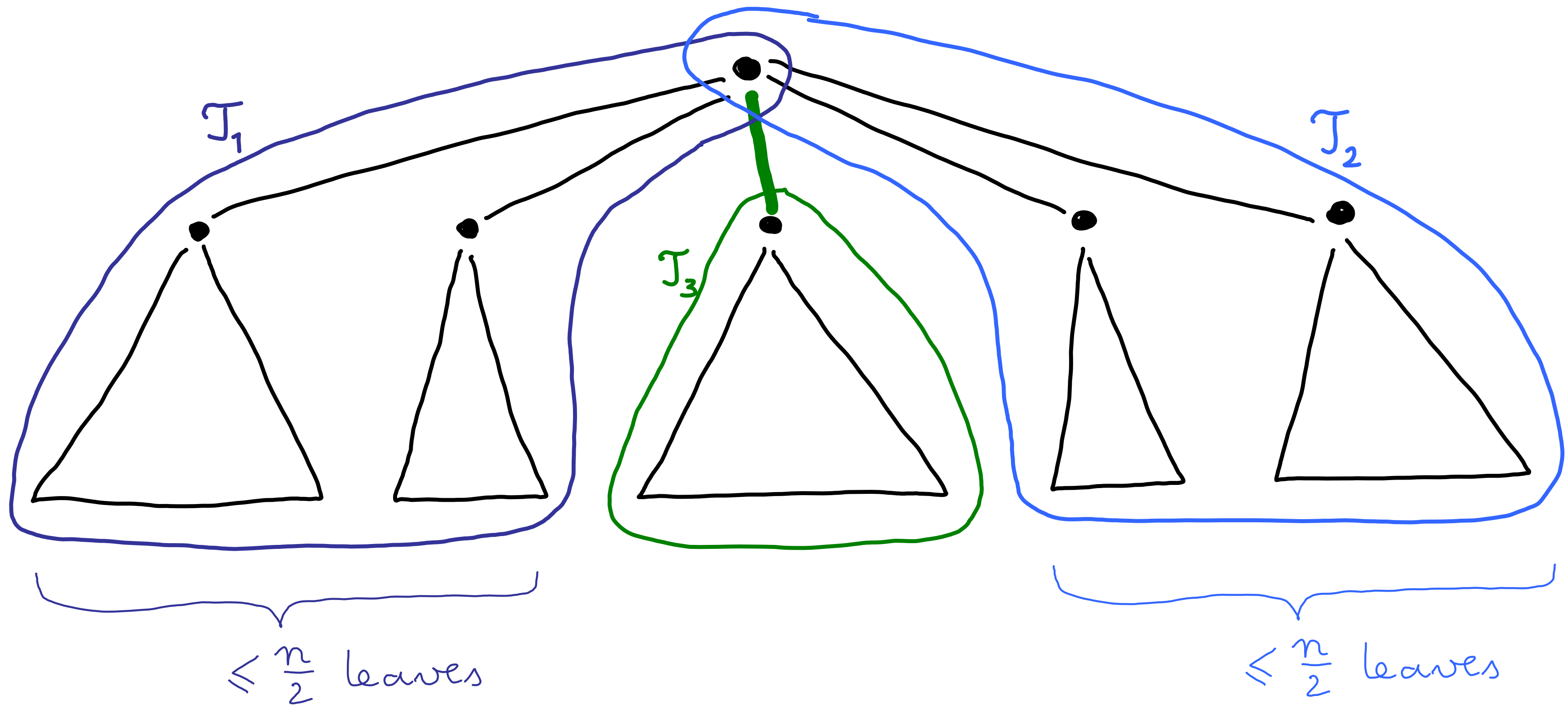
If an ordered tree T has height $\frac{d}{2}$ and $\leq n$ leaves then there is an order-preserving labelling of T by $(\{0,1\}^*, <)$ s.t. on every path (from root to leaf) $\leq \lg n$ bits are used



TREE CODING LEMMA

LEMMA

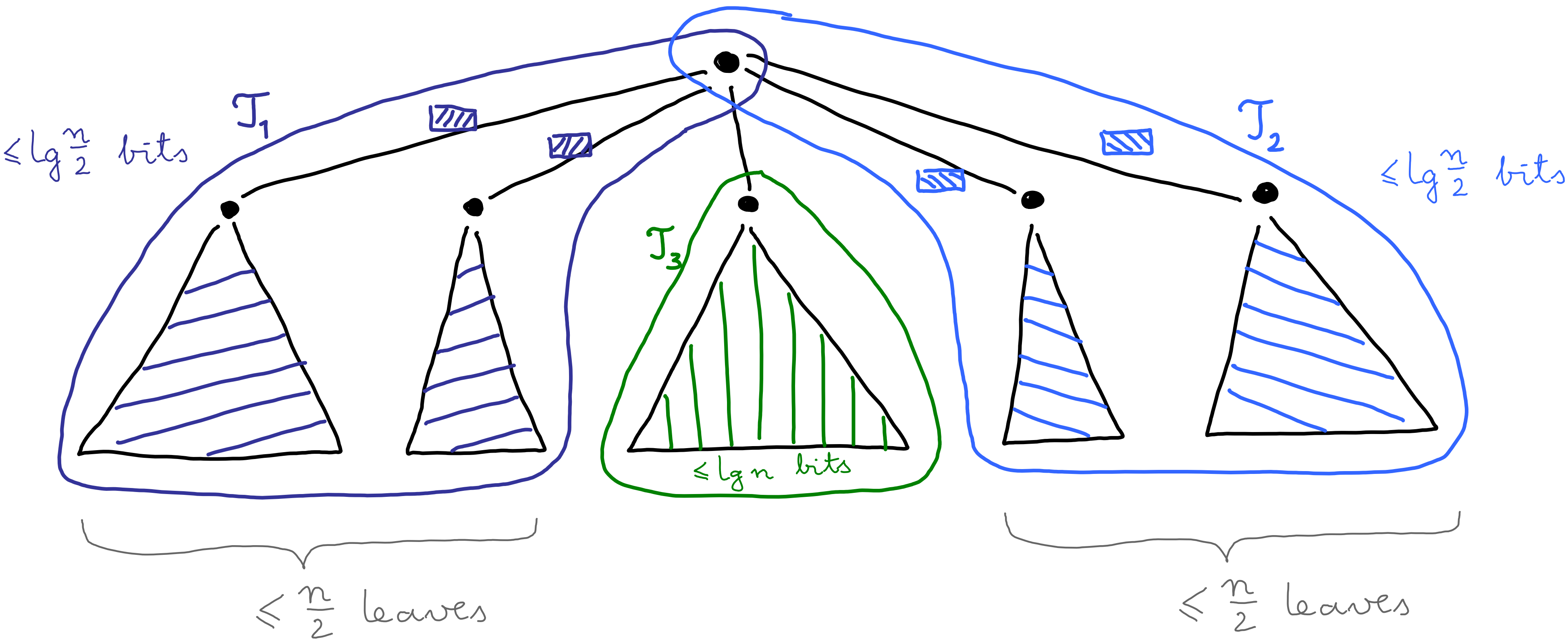
If an ordered tree T has height $\frac{d}{2}$ and $\leq n$ leaves then there is an order-preserving labelling of T by $(\{0,1\}^*, <)$ s.t. on every path (from root to leaf) $\leq \lg n$ bits are used



TREE CODING LEMMA

LEMMA

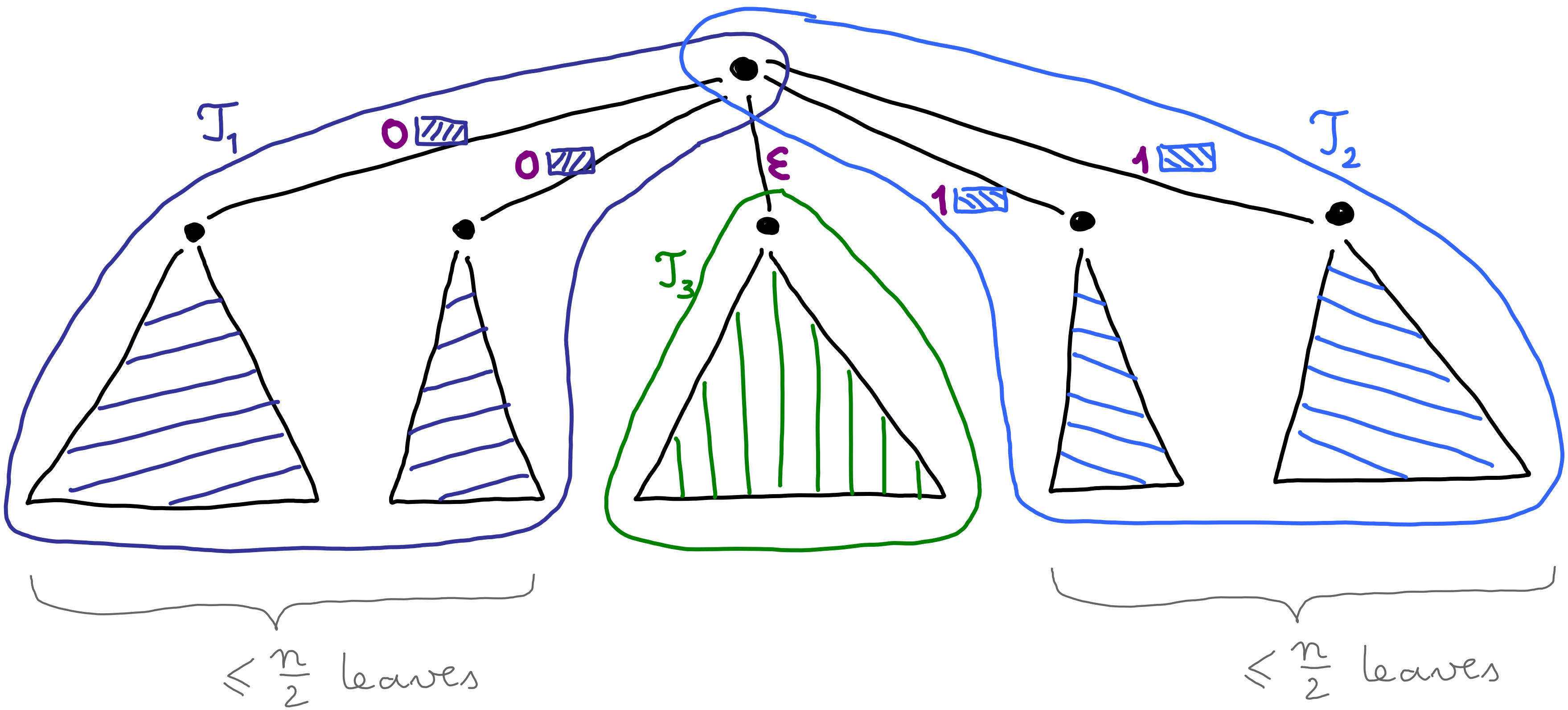
If an ordered tree T has height $\frac{d}{2}$ and $\leq n$ leaves then there is an order-preserving labelling of T by $(\{0,1\}^*, <)$ s.t. on every path (from root to leaf) $\leq \lg n$ bits are used



Tree Coding Lemma

Lemma

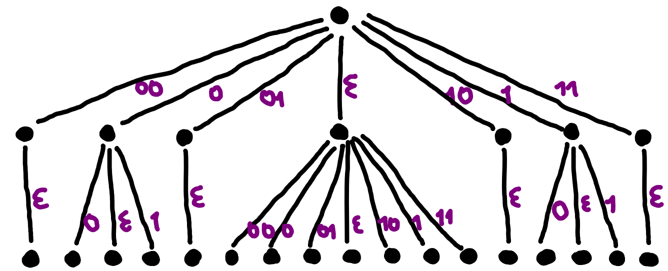
If an ordered tree T has height $\frac{d}{2}$ and $\leq n$ leaves then there is an order-preserving labelling of T by $(\{0,1\}^*, <)$ s.t. on every path (from root to leaf) $\leq \lg n$ bits are used



SUCCINCT MULTI-COUNTERS

$$S_{n,d} \stackrel{\text{df}}{=} \left\{ \langle s_{d-1}, s_{d-3}, \ldots, s_1 \rangle : \; s_p \in \{0,1\}^* \text{ and } \sum_{\text{odd } p} |s_p| \leqslant \lceil \lg n \rceil \right\}$$

$S_{4,4}$



$00 \prec 0 \prec 01 \prec \varepsilon \prec 10 \prec 1 \prec 11$

bits sufficient to represent a succinct multi-counter

Fact $$|S_{n,d}| \leqslant 2^{\overbrace{\lceil \lg n \rceil \cdot (1 + \lceil \lg \frac{d}{2} \rceil)}} = \boxed{n^{\lg d + O(1)}}$$

For each bit,

the bit

the co-ordinate the bit belongs to

THE SIZE OF $S_{n,d}$

- $$|S_{n,d}| \leqslant 2^{\lceil \lg n \rceil} \cdot \binom{\lceil \lg n \rceil + \frac{d}{2}}{\frac{d}{2}}$$

- $$|S_{n,d}| = \begin{cases} O\left(n \cdot \lg^{d/2} n\right) & \text{if } d = O(1) \\ O\left(n^{1+o(1)}\right) & \text{if } d = o(\log n) \\ \widetilde{\Theta}\left(n^{\lg(\delta+1) + \underbrace{\lg(e_\delta)+1}_{\leqslant 3.72}}\right) & \text{if } d = 2\lceil \delta \cdot \lg n \rceil \\ O\left(d \cdot n^{\lg\left(\frac{d}{\lg n}\right) + 1.45}\right) & \text{if } d = \omega(\log n) \end{cases}$$

where $e_\delta = \left(1 + \frac{1}{\delta}\right)^\delta$

UNIVERSAL TREES AND SEPARATING AUTOMATA ARE QUASI-POLYNOMIAL

THEOREM [Czerwiński, Daviaud, Fijalkow, J., Lazić, Parys 2018]

The sizes of smallest universal trees and of smallest separating automata are quasi-polynomial

UNIVERSAL TREES ≡ SEPARATING AUTOMATA

THEOREM [Czerwiński, Daviaud, Fijalkow, J., Lazić, Parys 2018]

Leaves of every $(n, \frac{d}{2})$-universal tree are the states of a strong (n, d)-separator

THEOREM [Czerwiński, Daviaud, Fijalkow, J., Lazić, Parys 2018]

States in every strong (n, d)-separator include all the leaves in an $(n, \frac{d}{2})$-universal tree

SMALLEST UNIVERSAL TREES ARE QUASI-POLYNOMIAL

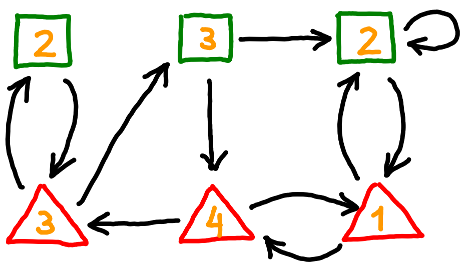
THEOREM [J., Lazić 2017]

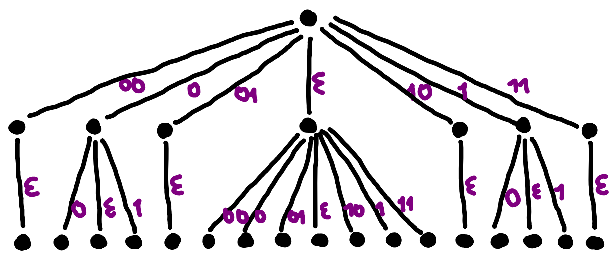
There is an $(n, \frac{d}{2})$-universal tree of size $n \cdot \binom{\lg n + \frac{d}{2}}{\lg n} = n^{\lg\left(\frac{d}{\lg n}\right) + O(1)}$

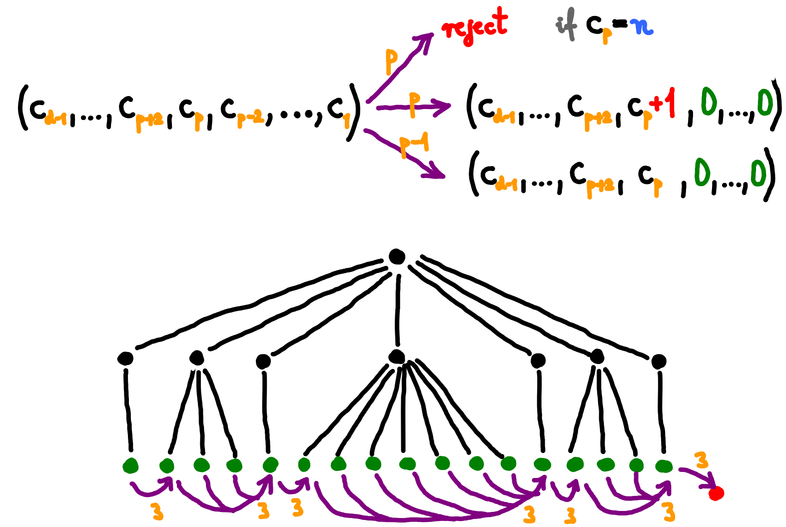
THEOREM [Czerwiński, Daviaud, Fijalkow, J., Lazić, Parys 2018]

Every $(n, \frac{d}{2})$-universal tree is of size at least $\binom{\lg n + \frac{d}{2} - 2}{\lg n - 1} \geqslant n^{\lg\left(\frac{d}{\lg n}\right) - 2}$

PARITY GAMES, UNIVERSAL TREES, AND SEPARATING AUTOMATA







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