Knapsack Problems in Hyperbolic Groups

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Our setting

- Let G be a finitely generated (f.g.) group.
- Fix a finite generating set Σ for G with $a \in \Sigma \Leftrightarrow a^{-1} \in \Sigma$.
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- INPUT: Group elements $g, g_1, g_2, \ldots, g_k \in G$
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Decidability/complexity of knapsack does not depend on the chosen generating set for G.

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Easier than knapsack:
Replace g^x (with $x \in \mathbb{Z}$) by $g^{x_1}(g^{-1})^{x_2}$ (with $x_1, x_2 \in \mathbb{N}$).

Knapsack over $\mathbb{Z}$

The classical knapsack problem

- INPUT: Integers $a, a_1, \ldots a_k \in \mathbb{Z}$
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This problem is known to be decidable and the complexity depends on the encoding of the integers $a, a_1, \dots a_k \in \mathbb{Z}$:

- Binary encoding of integers (e.g. $5 \mathrel{\widehat{=}} 101$): NP-complete
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Note: Our definition of knapsack corresponds to the unary variant.

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In compressed knapsack the group elements $g, g_1, \ldots, g_k$ are encoded by SLPs that produce words over Σ.

Some known results

Knapsack is decidable for

- all virtually special groups
  $=$ finite extensions of subgroups of right-angled Artin groups
- all co-context-free groups
  $=$ groups where complement of word problem is context-free
- all Baumslag-Solitar groups $\mathrm{BS}(1, q) = \langle a, t \mid t^{-1}at = a^q \rangle$
- the discrete Heisenberg group $H_3(\mathbb{Z})$

Knapsack is undecidable for

- $H_3(\mathbb{Z})^k$ where k is a fixed large enough number.

Cayley graph

The Cayley graph $\Gamma = \Gamma(G, \Sigma)$ of G (w.r.t. Σ) is the graph with

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Geodesic triangles and slim triangles

A geodesic triangle Δ consists of points $p, q, r \in G$ and paths $P_{p,q}$, $P_{p,r}$, $P_{q,r}$ (the sides of the triangle), where $P_{x,y}$ is a path between x and y of length $d_\Gamma(x, y)$ (a geodesic path).

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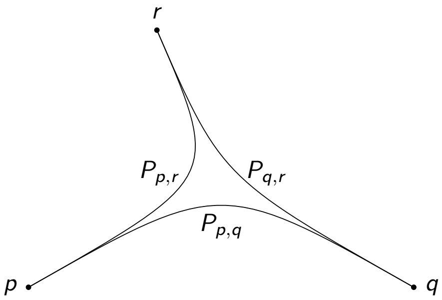
Δ is δ-slim for $\delta \geq 0$ if every point on a side $P_{x,y}$ has distance at most δ from a point belonging to one of the two opposite sides.

Hyperbolic groups (Gromov 1987)

A group is hyperbolic if there is a constant δ such that every geodesic triangle is δ-slim.

The shape of a geodesic triangle in a hyperbolic group:



Some facts about hyperbolic groups

- Let G be hyperbolic. Then, either
  1. $F_2 \leq G$ (nonelementary hyperbolic groups) or
  2. $\mathbb{Z} \leq G$ with $[G : \mathbb{Z}]$ finite (elementary hyperbolic groups)

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- Let G be hyperbolic. Then, either
  1. $F_2 \leq G$ (nonelementary hyperbolic groups) or
  2. $\mathbb{Z} \leq G$ with $[G : \mathbb{Z}]$ finite (elementary hyperbolic groups)
- The word problem for a hyperbolic group can be solved in
  1. linear time and
  2. belongs to the complexity class LogCFL $\subseteq \mathsf{NC}^2$.

  LogCFL = closure of context-free languages under logspace reductions.

Complexity of knapsack in hyperbolic groups

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Theorem 1

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- LogCFL-complete if G is nonelementary.

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Theorem 2

For every infinite hyperbolic group, compressed knapsack is NP-complete.

Some proof ingredients

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Let G be hyperbolic, $g, g_1, \dots, g_k$, and $N = |g| + |g_1| + \cdots + |g_k|$.

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Holt, L, Schleimer 2018

The compressed word problem for a hyperbolic group belongs to P.

Hyperbolic groups are knapsack-semilinear

(Semi-)linear sets

A subset $A \subseteq \mathbb{N}^k$ is linear if there exist $v_0, v_1, \ldots, v_n \in \mathbb{N}^k$ such that

$$A = \{v_0 + \lambda_1 v_1 + \cdots + \lambda_n v_n \mid \lambda_1, \ldots, \lambda_n \in \mathbb{N}\}.$$

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Theorem 3

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- Knapsack for automaton groups:

  There are automaton groups with undecidable knapsack problem (powers of Heisenberg group).

  For which automaton groups is knapsack decidable?