Büchi VASS recognise Σ^1_1-complete ω-languages

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Reachability Problems 2018

Marseille 26.09.2018

Foundation for Polish Science

UNIVERSITY OF WARSAW

NATIONAL SCIENCE CENTRE POLAND

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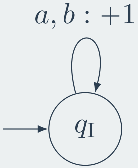

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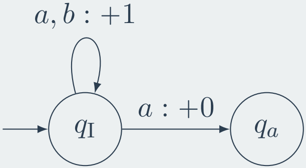

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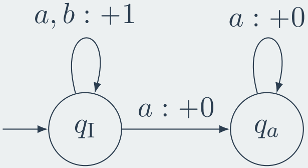

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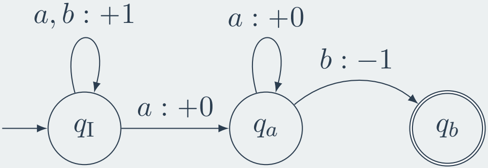

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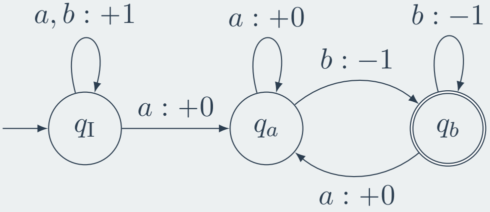

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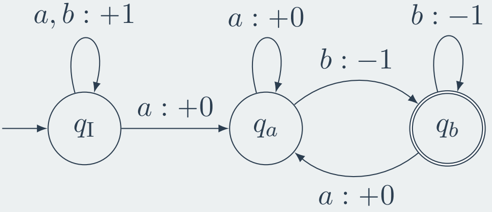

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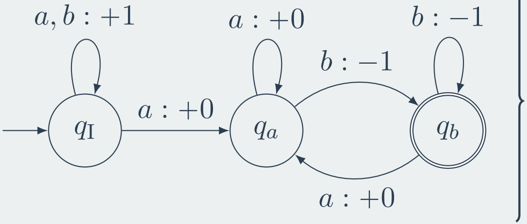

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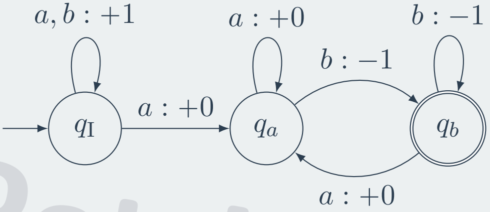

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Corollary

If $C = \{c\}$ then each anti-chain in $\big(\mathrm{Conf}, \preceq\big)$ is $\leqslant |Q|$.

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Btw.: all standard acceptance conditions are **Borel**.

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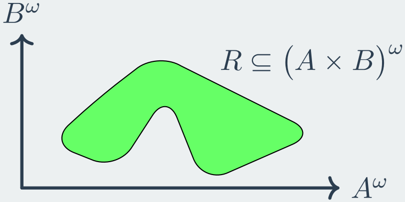

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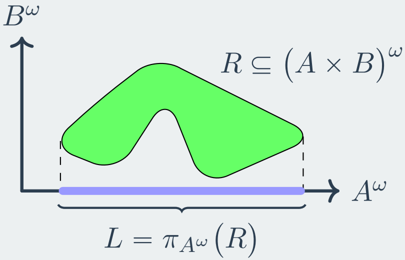

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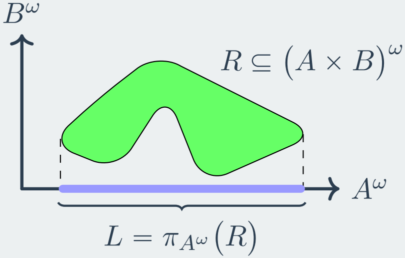

Analytic (Σ^1_1) sets $\overset{\text{def}}{\equiv}$ projections of **Borel** sets

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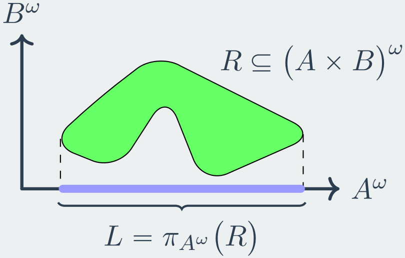

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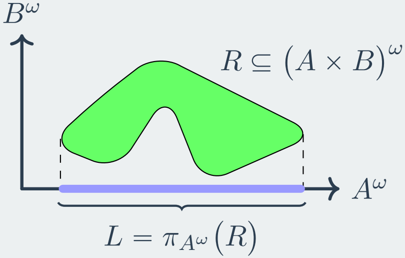

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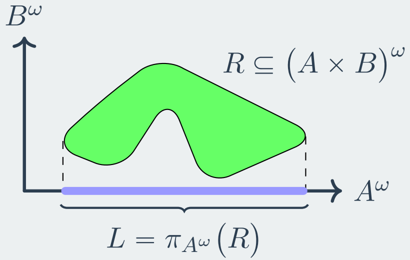

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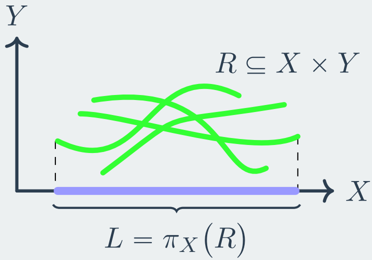

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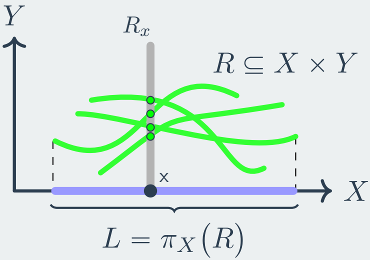

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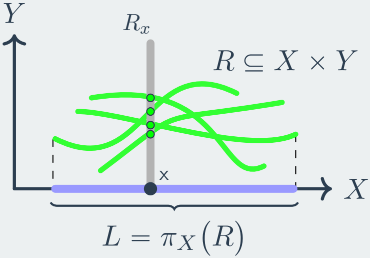

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$\{y \in Y \mid (x, y) \in R\}$ // R_x

If $R \subseteq X \times Y$ is **Borel** and $\forall x \in X.\ |R_x| \leqslant \aleph_0$ then $\pi_X(R)$ is **Borel**.

Weak Non-determinism vs. Full Non-determinism

Weak Non-determinism vs. Full Non-determinism

$\left(\text{deterministic or countably unambiguous machines}\right)$ $\left(\text{models with inherent guess-based choices}\right)$

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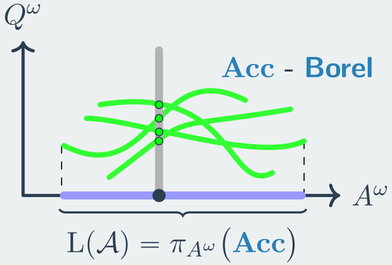

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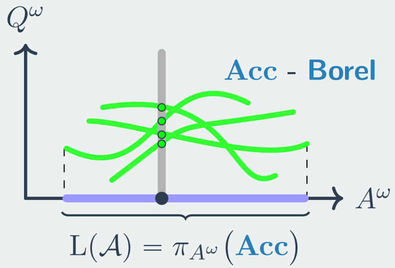

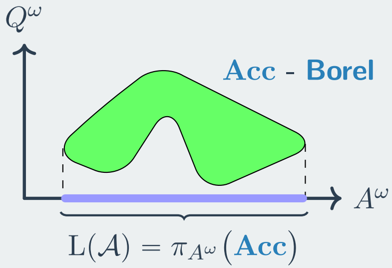

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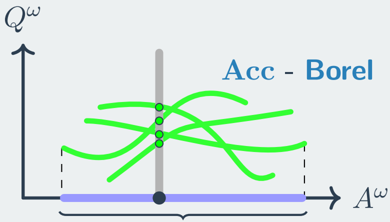

$L(\mathcal{A}) = \pi_{A^\omega}(\mathbf{Acc})$

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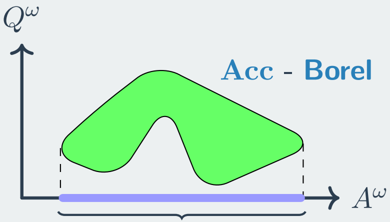

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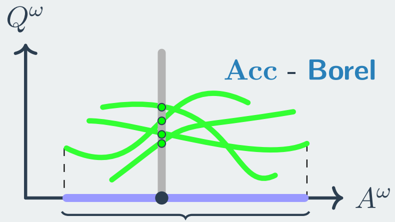

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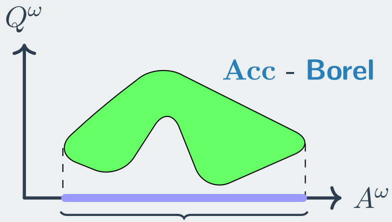

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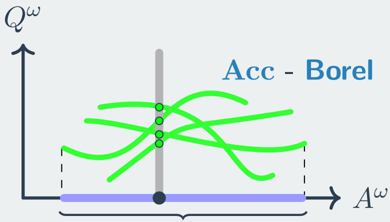

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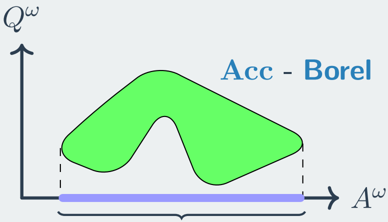

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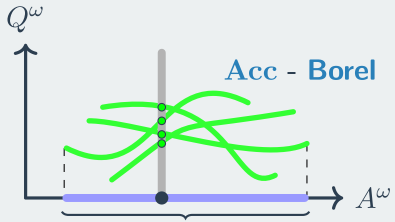

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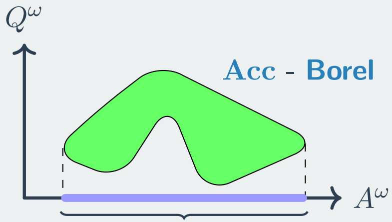

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There exists a **non Borel** ω-language

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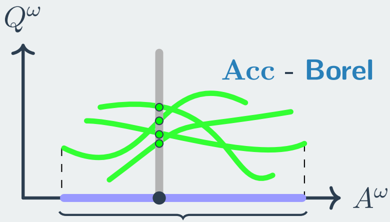

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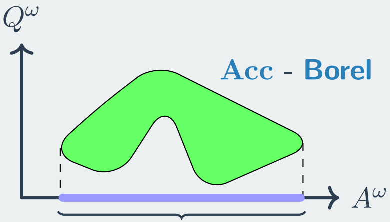

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There exists a **non Borel** ω-language recognisable by a one-counter Büchi VASS.

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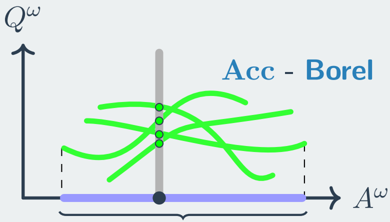

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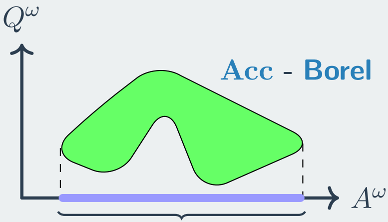

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Analytic-complete

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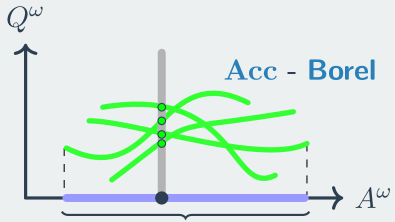

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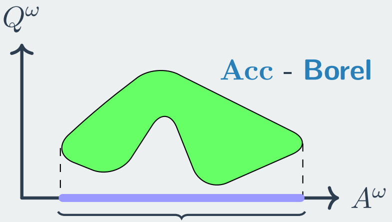

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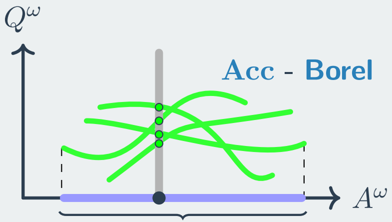

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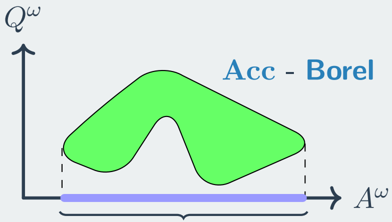

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(**Analytic**-complete)

$\longrightarrow$ Similar result (+Wadge) with four counters in (Finkel ['18]) $\longleftarrow$

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Examples of $\mathbf{\Sigma}_1^1$-complete (and thus **non Borel**) sets

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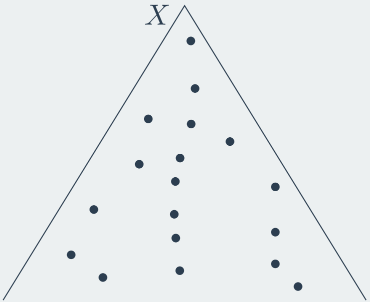

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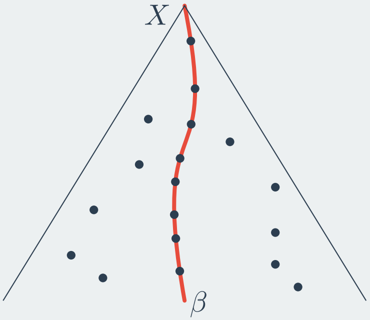

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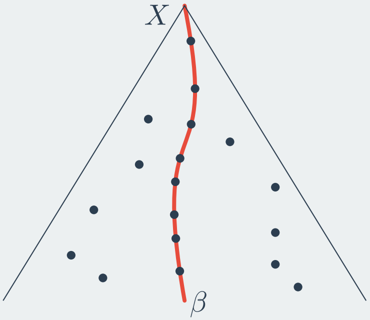

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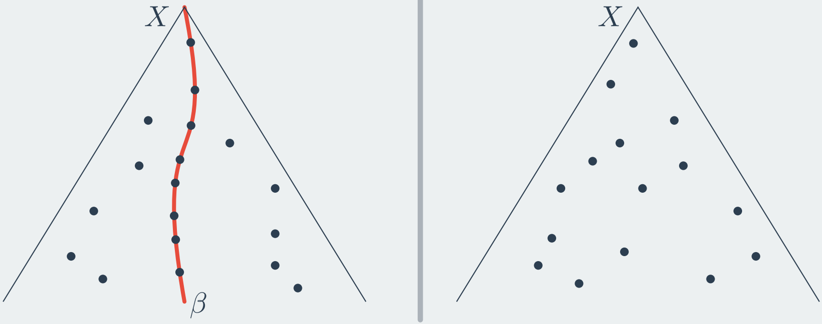

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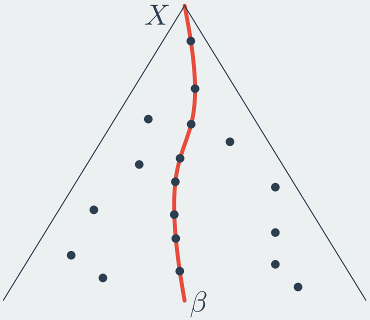

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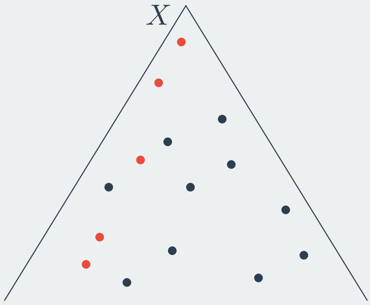

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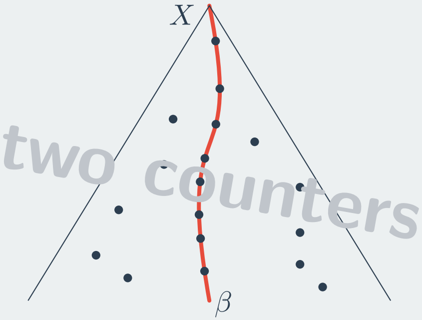

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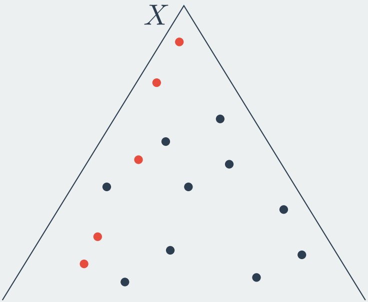

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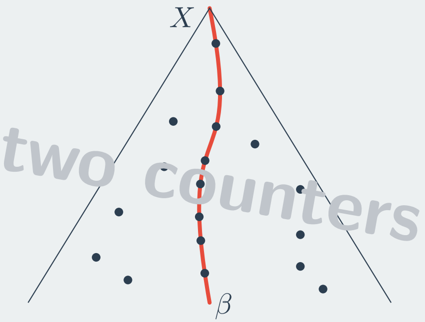

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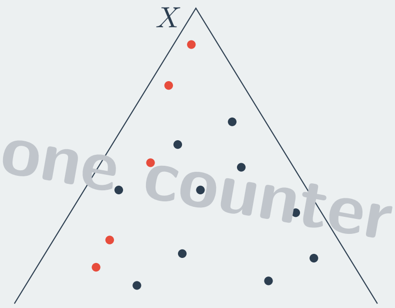

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Values to store: u_0, u_1, ..., u_{N-1}.

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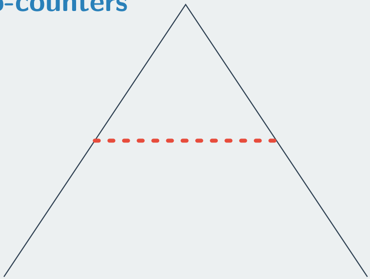

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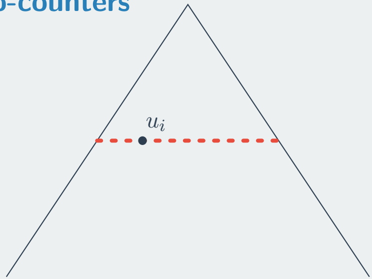

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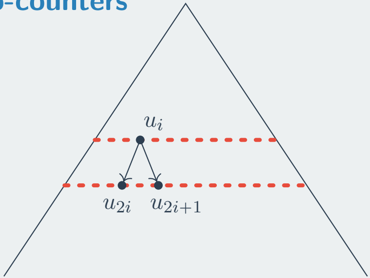

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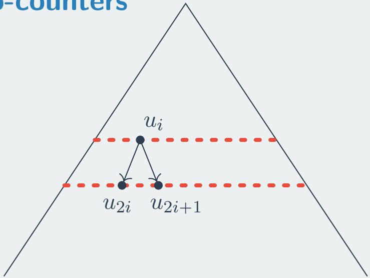

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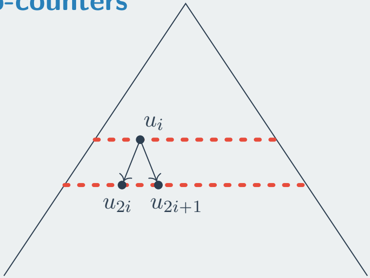
$$\downarrow$$



Encoding in two-counters

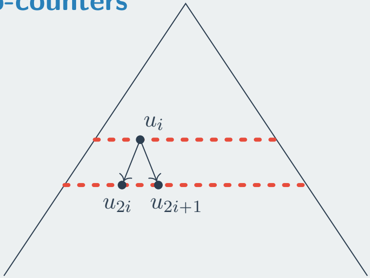
Values to store: u_0, u_1, ..., u_{N-1}.

$$u_i \mapsto (i, N-i)$$

$$u_i \longrightarrow u_{i'}$$

$$\downarrow$$

$$\big(-i, -(N-i)\big),\ \big(+i', +(N'-i')\big)$$



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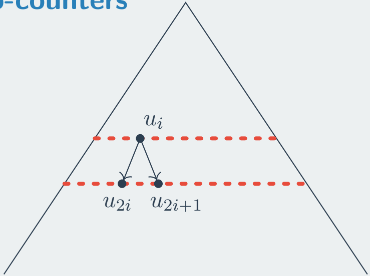

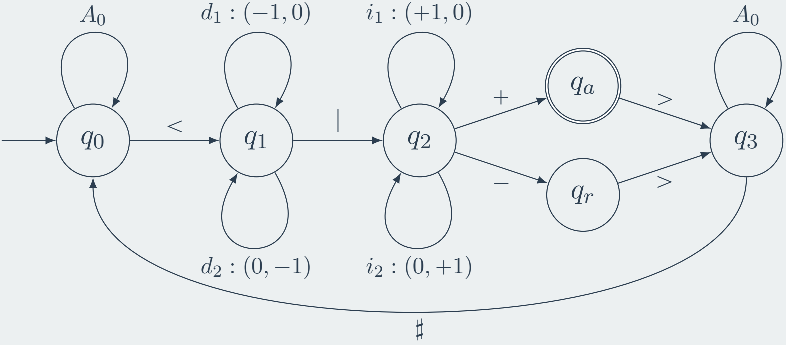

Encoding in one-counter

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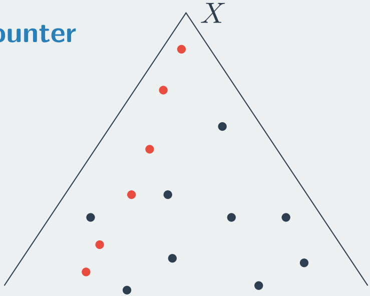

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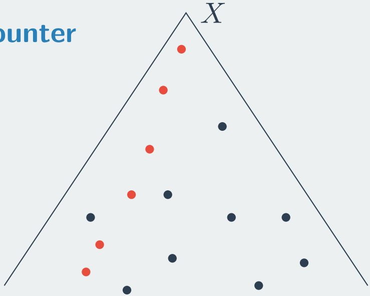

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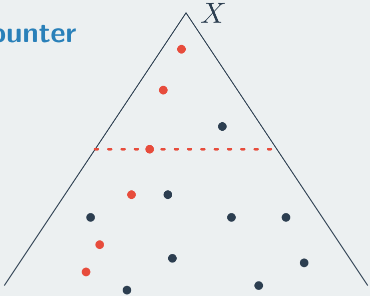

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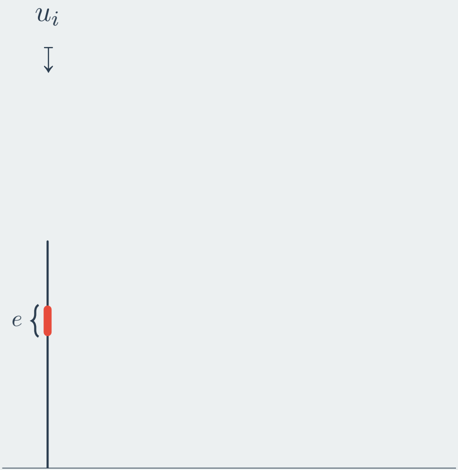

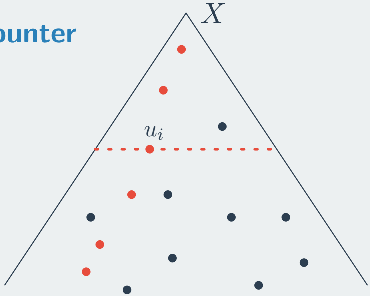

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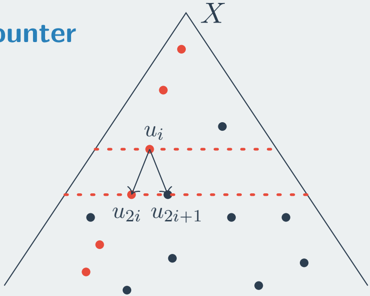

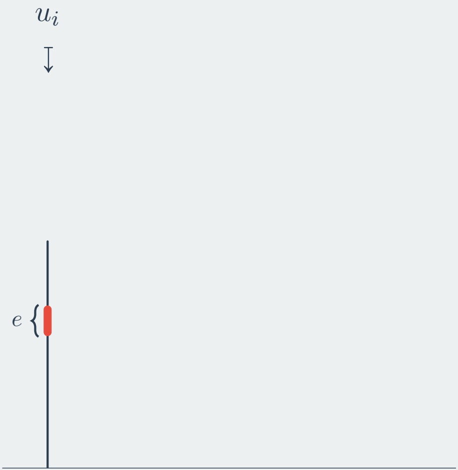

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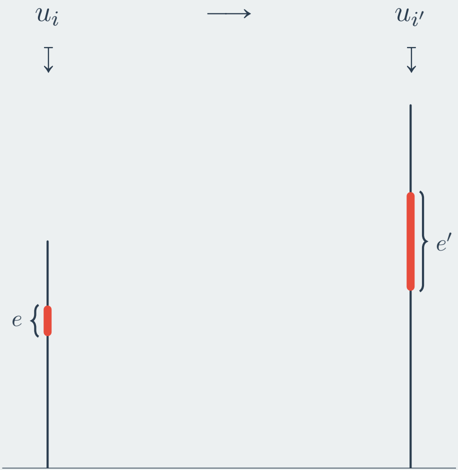

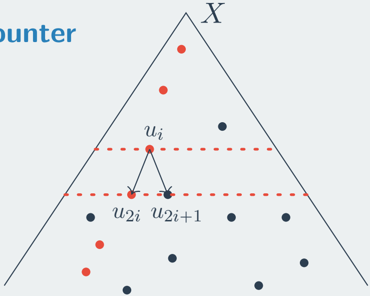

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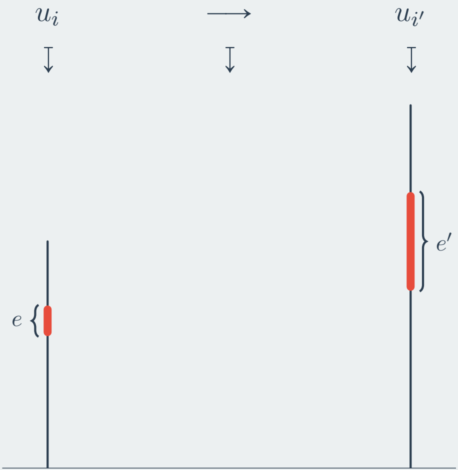

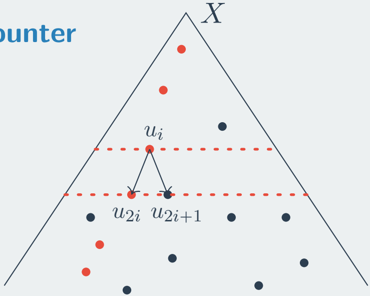

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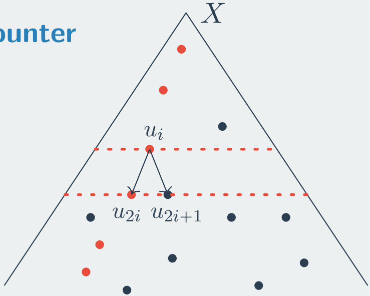

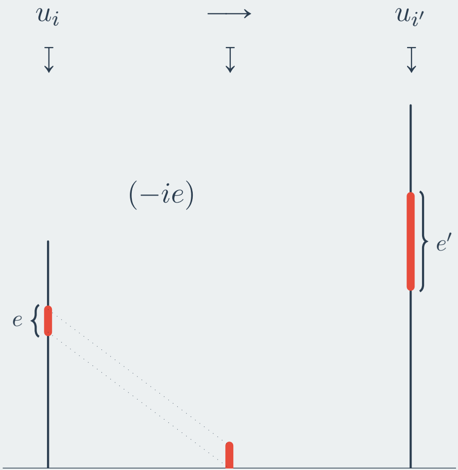

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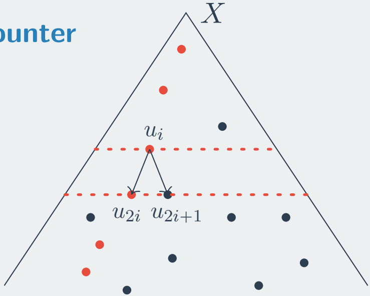

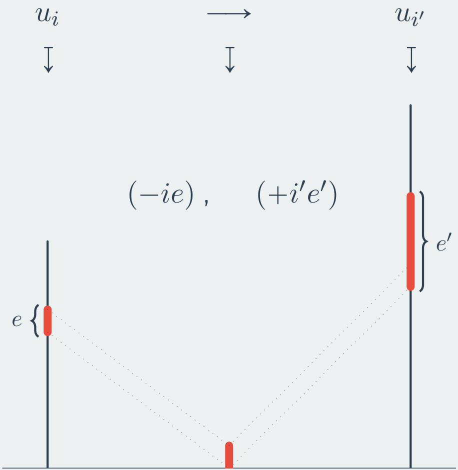

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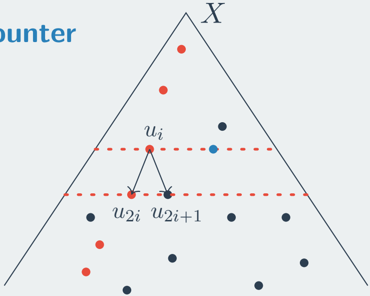

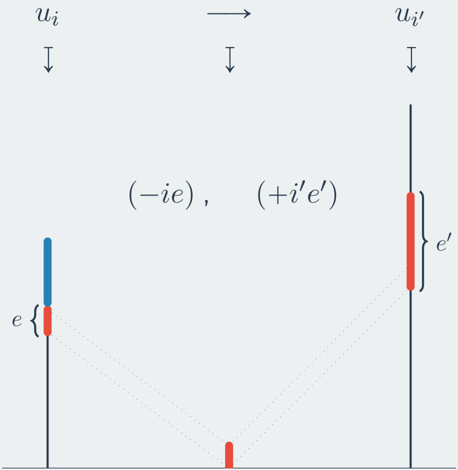

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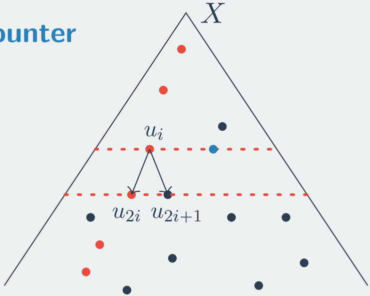

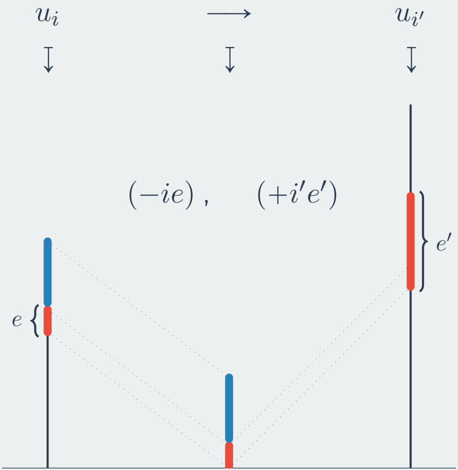

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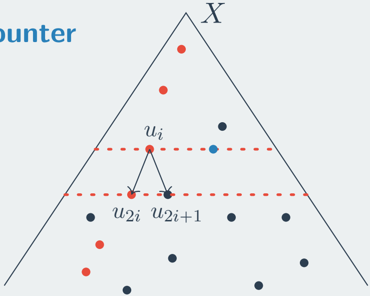

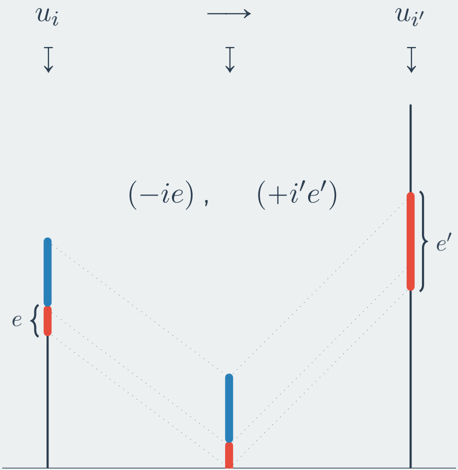

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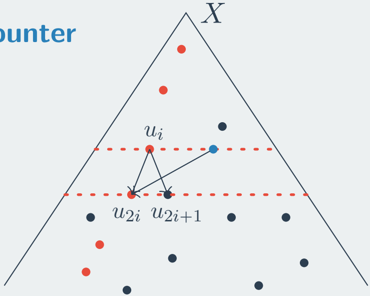

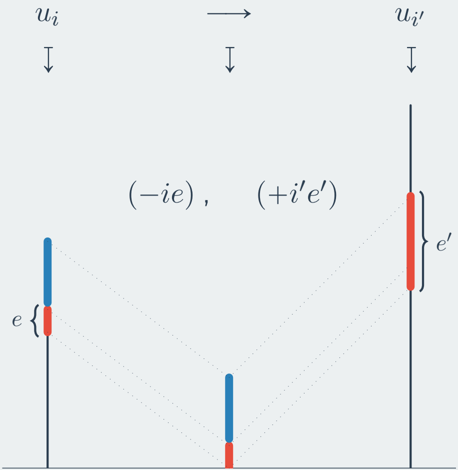

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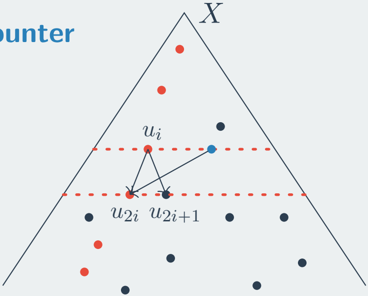

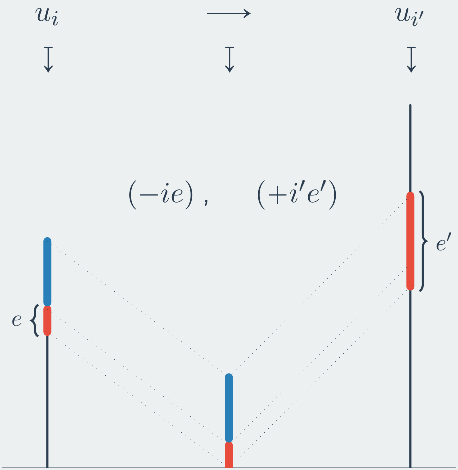

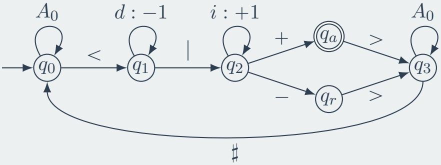

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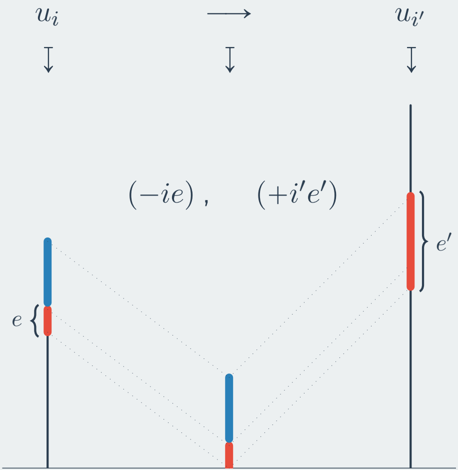
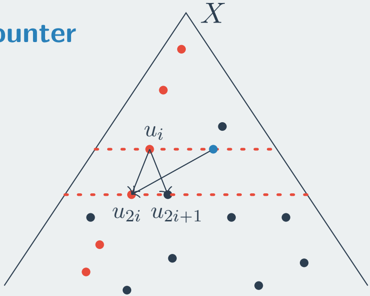
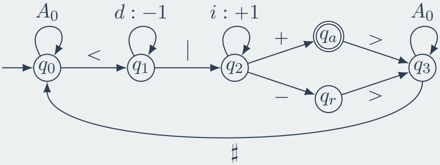




$$f\colon X \mapsto \alpha \in A^\omega$$



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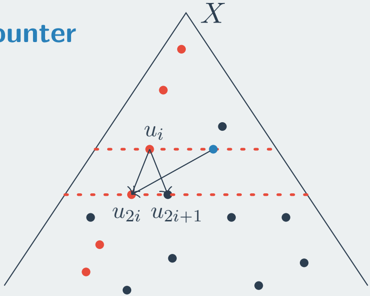
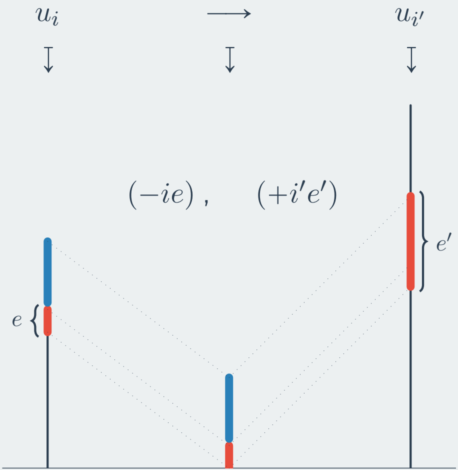
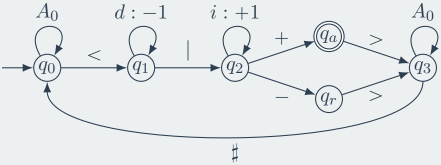




$$f\colon X \mapsto \alpha \in A^\omega$$

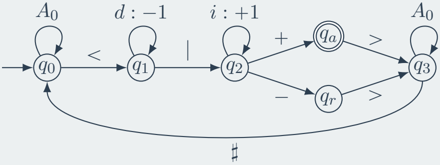
X is $\leqslant_{\text{prefix}}$-**IF** $\iff$ $f(X) \in \mathrm{L}(\mathcal{A})$



Summary

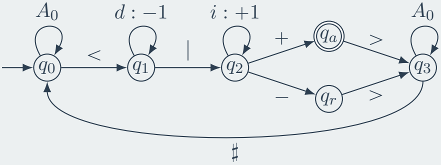
Summary

Simple automaton



Summary

Simple automaton



Simple proof

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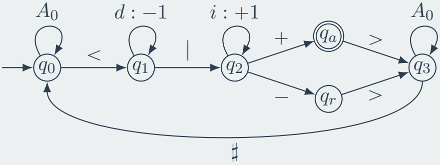
X is $\leqslant_{\mathrm{prefix}}$-**IF** $\iff f(X) \in \mathrm{L}(\mathcal{A})$

Summary

Simple automaton



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Full non-determinism