Reachability Problems in Nondeterministic Polynomial Maps on the Integers

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Reachability in Iterative Maps

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$$\overline{\mathbf{x}}_{n+1} = f(\overline{\mathbf{x}}_n)$$

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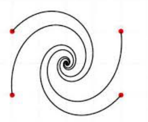
$$\exists k \in \mathbb{N}, \qquad \bar{\mathbf{y}} = f^k(\bar{\mathbf{x}}_0)$$



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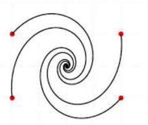

- The complexity could vary depending on the factors such as
  - the type of iterative functions (i.e., affine, linear, polynomial, elementary, etc.),
  - the form of maps (i.e., deterministic, nondeterministic),
  - the number of variables (i.e., dimension of a system), and
  - even history dependence (i.e., when the next value depends on several previous values of counters/variables).

Program Termination

Halting problem (Alan Turing, 1936)

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while (conditions) { commands }

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With two independent variables

Let's consider termination of the following simple program:

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\begin{aligned}
&\textbf{while}\ (\ x \neq t_1\ \textbf{and}\ y \neq t_2\ )\ \{ \\
&\qquad x \leftarrow ax^2 + bx + c \\
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\end{aligned}
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With dependency between variables

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With nondeterminism

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&\textbf{while}\ (\ x \neq t_1\ \textbf{and}\ y \neq t_2\ )\ \{ \\
&\qquad x \leftarrow ax^2 + by + c\ \textbf{or}\ x \leftarrow ax + b \\
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\end{aligned}
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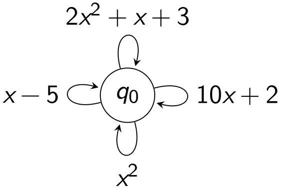
Nondeterministic Polynomial Map

Definition (Nondeterministic polynomial map)

An *n-dimensional (nondeterministic) polynomial map* is a tuple $\mathcal{R} = (Q, \Delta)$, where

- Q is a **singleton** set and
- $\Delta \subseteq \mathbb{Z}[\overline{\mathbf{x}}]^n$ is a finite set of transitions labelled by polynomials with variable $\overline{\mathbf{x}} \in \mathbb{Z}^n$.



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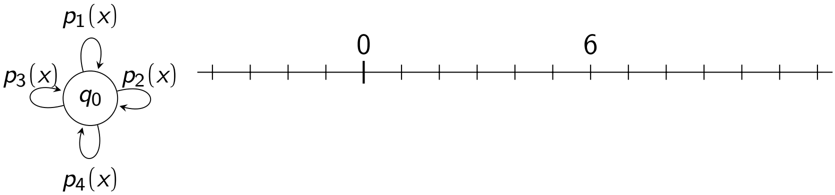

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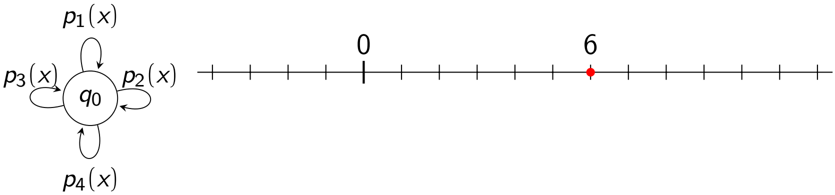

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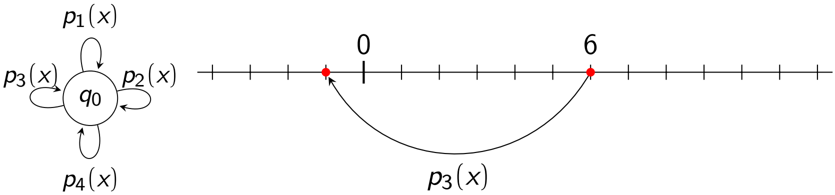

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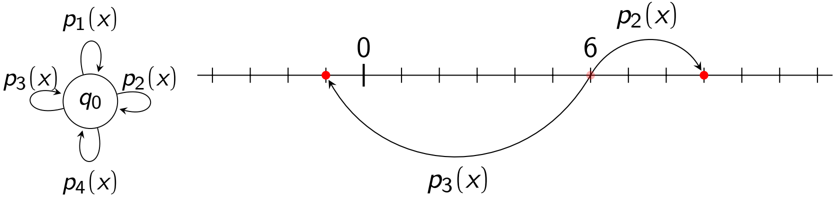

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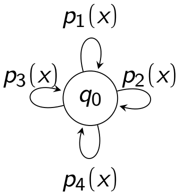

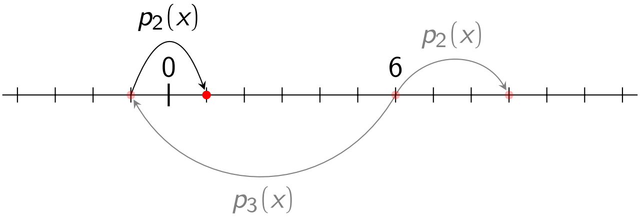

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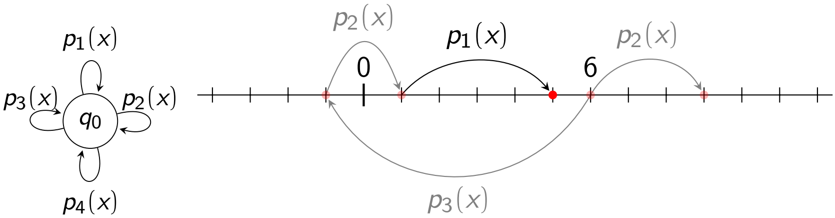

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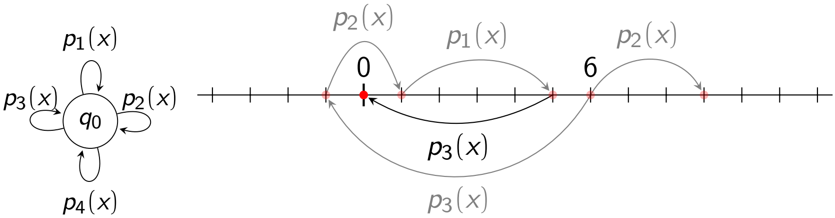
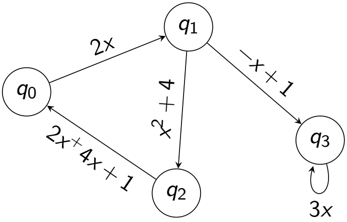


Polynomial Register Machine (PRM)

Definition (Polynomial register machine)

An *n-dimensional polynomial register machine* (n-PRM) is a tuple $\mathcal{R} = (Q, \Delta)$, where

- Q is a finite set of states and
- $\Delta \subseteq Q \times \mathbb{Z}[\overline{\mathbf{x}}]^n \times Q$ is a finite set of transitions labelled by polynomials with variable $\overline{\mathbf{x}} \in \mathbb{Z}^n$.



Class of Polynomials

Definition

Additive polynomials: $\mathrm{Add}_{\mathbb{Z}} = \{\pm x + b \mid b \in \mathbb{Z}\}$,

Affine polynomials: $\mathrm{Aff}_{\mathbb{Z}}[x] = \{ax + b \mid a, b \in \mathbb{Z}\}$,

Quadratic polynomials: $\mathrm{Quad}_{\mathbb{Z}}[x] = \{ax^2 + bx + c \mid a, b, c \in \mathbb{Z}\}$.

$$a_n x^n + \ldots + \overbrace{a_2 x^2 + \underbrace{a_1 x + a_0}_{\in \mathrm{Aff}_{\mathbb{Z}}[x]}}^{\in \mathrm{Quad}_{\mathbb{Z}}[x]} \in \mathbb{Z}[x] \qquad \pm x + a_0 \in \mathrm{Add}_{\mathbb{Z}}$$

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Definition (Polynomials without additive polynomials)

$$\mathrm{Aff}_{\mathbb{Z}}[x] \setminus \mathrm{Add}_{\mathbb{Z}} = \{ax + b \in \mathrm{Aff}_{\mathbb{Z}}[x] \mid a \neq \pm 1\},$$

$$\mathbb{Z}[x] \setminus \mathrm{Add}_{\mathbb{Z}} = \{p(x) \in \mathbb{Z}[x] \mid p(x) \neq \pm x + b, \text{ where } b \in \mathbb{Z}\}.$$

Previous Work

- The **additive** form (i.e., $\bar{\mathbf{x}} \leftarrow \bar{\mathbf{x}} + \bar{\mathbf{b}}$) of a map with polynomial updates can be seen as a vector addition systems on $\mathbb{Z}^n$.
  - If $n = 1$, the reachability problem can be reduced to the solution of a single linear Diophantine equation over natural numbers.
  - Otherwise, the problem is in the form of the n-dimensional VAS on $\mathbb{Z}^n$.

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  - Otherwise, the problem is in the form of the n-dimensional VAS on $\mathbb{Z}^n$.

- Bell & Potapov showed that with **seven** 2-d affine updates of the form

$$\begin{cases} x \leftarrow ax + by + c \\ y \leftarrow dy + e \end{cases}, \quad \text{(variables are not independent, stateless)}$$

the reachability problem is undecidable over $\mathbb{Q}^2$. [TCS 2008]

Previous Work

- Finkel et al. [MFCS 2013] considered that the reachability problem for polynomial register machines (with states) on $\mathbb{Z}^n$,
  - PSPACE-complete for 1-d polynomials and
  - undecidable for 2-d polynomials with independent variables.

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- Niskanen [RP 2017] showed that the reachability problem is
  - PSPACE-complete in 1-d polynomial maps of degree four and
  - undecidable in 3-d polynomial maps. (stateless)

Reachability in Maps over $\mathrm{Aff}_{\mathbb{Z}}[x]^3$ and $\mathrm{Quad}_{\mathbb{Z}}[\overline{\mathbf{x}}]^2$

In the three-dimensional variant, we are investigating functions of the form

$$\begin{cases} x_1 \leftarrow a_1 x_1 + b_1 \\ x_2 \leftarrow a_2 x_2 + b_2 \\ x_3 \leftarrow a_3 x_3 + b_3 \end{cases}, \quad \text{where} \quad a_i, b_i \in \mathbb{Z}.$$

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First, we will show that

- The reachability problem for $\mathrm{Aff}_{\mathbb{Z}}[\overline{\mathbf{x}}]^3$ is undecidable and
- PSPACE-hard for $\mathrm{Quad}_{\mathbb{Z}}[\overline{\mathbf{x}}]^2$.

Undecidability over $\mathrm{Aff}_{\mathbb{Z}}[\overline{\mathbf{x}}]^3$

Theorem

The reachability problem for maps over $\mathrm{Aff}_{\mathbb{Z}}[\overline{\mathbf{x}}]^3$ is undecidable with at least 7 affine functions over $\mathbb{Z}$.

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Proof sketch.

- Let $P = \{(u_1, v_1), \ldots, (u_n, v_n)\} \subseteq \Sigma^* \times \Sigma^*$ be an instance of the PCP.

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	(u_1, v_1)	(u_2, v_2)	(u_1, v_1)	(u_3, v_3)
$u =$	221	12	221	1
$v =$	22	11	22	211

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- We can simulate concatenations with affine functions as follows:

$$3^{|u_i|}\sigma(u_j) + \sigma(u_i) = \sigma(u_j u_i).$$

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- Define the following sets of affine functions in dimension three:
  - $F_1 = \{(3^{|u_i|}x_1 + \sigma(u_i), 3^{|v_i|}x_2 + \sigma(v_i), 2x_3) \mid (u_i, v_i) \in P \text{ for all } 1 \leqslant i \leqslant n\}$,
  - $F_2 = \{(3^{|u_i|}x_1 + \sigma(u_i), 3^{|v_i|}x_2 + \sigma(v_i), 2x_3 + 1) \mid (u_i, v_i) \in P \text{ for all } 1 \leqslant i \leqslant n\}$,
  - $F_3 = \{(x_1 - 1, x_2 - 1, 2x_3 - 1)\}$.

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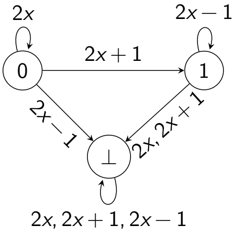

Simulating State Structure with Affine Functions

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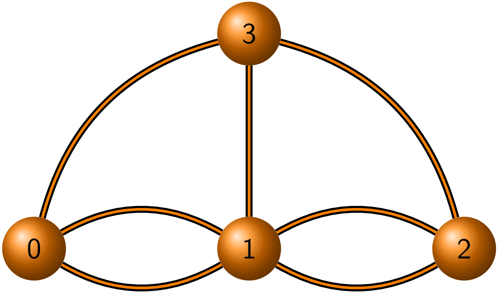

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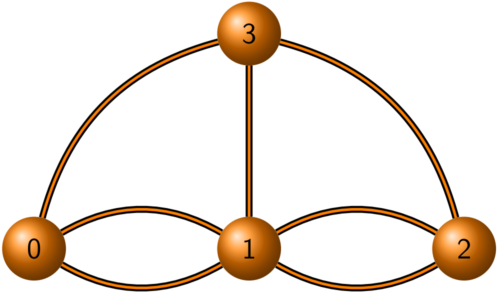

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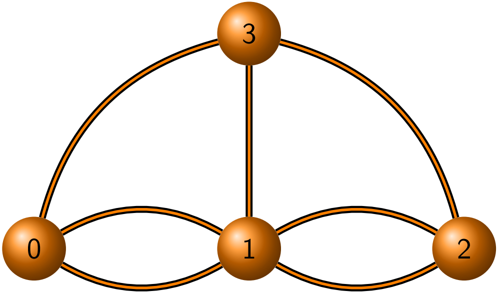

- For each edge (v_i, v_j) (possibly $i = j$) of G, we add an affine polynomial $f_{ij}(x) = m(x - i) + j$ to the map.
- For example, let us try with $f_{03}(x) = 4x + 3$ and $f_{21}(x) = 4(x - 2) + 1$.

PSPACE-hard over $\text{Quad}_{\mathbb{Z}}[\overline{\mathbf{x}}]^2$

Theorem

The reachability problem for nondeterministic maps over $\text{Quad}_{\mathbb{Z}}[\overline{\mathbf{x}}]^2$ is PSPACE*-hard.*

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- Note that the update polynomials of $\mathcal{R}$ are **quadratic**.
- For each transition $(q_i, p(x), q_j)$ of $\mathcal{R}$, we add two-dimensional function $(p(x), m \cdot x + j - m \cdot i)$ to the map.

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- For each transition $(q_i, p(x), q_j)$ of $\mathcal{R}$, we add two-dimensional function $(p(x), m \cdot x + j - m \cdot i)$ to the map.
- It is clear that $(0, k)$ is reachable from $(0, \ell)$ if and only if $[q_\ell, 0] \to^*_{\mathcal{R}} [q_k, 0]$.

What happens without additive updates?

- Let's consider a restricted class of maps over $\mathrm{Aff}_{\mathbb{Z}}[x]$, in the sense that every affine function in the map is not of the form $\pm x + b$.

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Question

Does the NP-hardness still hold over the restricted class of maps over $\mathrm{Aff}_{\mathbb{Z}}[x] \setminus \mathrm{Add}_{\mathbb{Z}}$?

NP-hardness over $\mathrm{Aff}_{\mathbb{Z}}[x] \setminus \mathrm{Add}_{\mathbb{Z}}$

Lemma

The reachability problem for maps over $\mathit{Aff}_{\mathbb{Z}}[x] \setminus \mathit{Add}_{\mathbb{Z}}$ is NP*-hard.*

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Proof sketch.

- Let (S, s) be an instance of the SSP, where $S = \{s_1, \ldots, s_k,\}$ and s is the target integer.

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$$F = \{n \cdot x + n^{i-1} \cdot s_i, \quad n \cdot x \mid 1 \leqslant i \leqslant k\}$$

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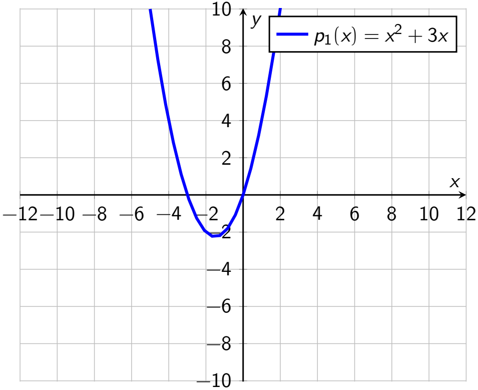
- The map reaches $s \cdot n^{k-1}$ if and only if there is a subset of S such that its elements add up to s.

PSPACE Upper Bound over $\mathbb{Z}[\overline{\mathbf{x}}]^n \setminus \mathrm{Add}_{\mathbb{Z}}$



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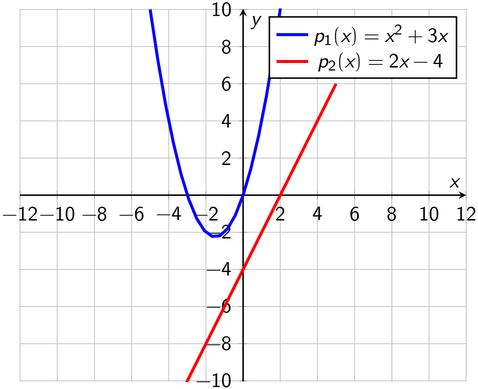

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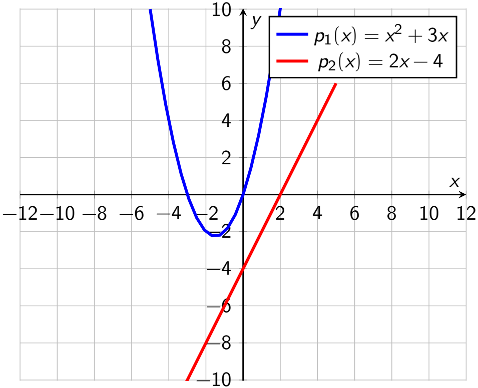

Observation

There exists a bound $b \in \mathbb{N}$ such that every polynomial in $\mathbb{Z}[x] \setminus \mathrm{Add}_{\mathbb{Z}}$ is monotonically increasing or decreasing in $\mathbb{Z} \setminus [-b, b]$.

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Theorem

The reachability problem for maps over $\mathbb{Z}[\overline{\mathbf{x}}]^n \setminus \mathsf{Add}_{\mathbb{Z}}$ is decidable in PSPACE for any $n \geqslant 1$.

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- We can compute the bound b which is polynomial in size of the input.
- If $|z| \leqslant b$, we can decide whether the integer z is reachable in PSPACE by applying the given functions since we can store the current value and the computation path in space polynomial in b.
- Otherwise, due to monotonicity properties of $\mathbb{Z}[x] \setminus \mathrm{Add}_{\mathbb{Z}}$ functions, we do not need to consider the integers outside the interval $[-z, z]$.

Undecidability over $\mathrm{Aff}_{\mathbb{Q}}[\overline{\mathbf{x}}]^3 \setminus \mathrm{Add}_{\mathbb{Q}}$

Theorem

The reachability problem for nondeterministic maps over $\mathrm{Aff}_{\mathbb{Q}}[\overline{\mathbf{x}}]^3 \setminus \mathrm{Add}_{\mathbb{Q}}$ *is undecidable with at least* 11 *affine functions over* $\mathbb{Q}$.

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Proof sketch.

- Let P be an instance of the PCP with n elements.
- For each pair $(u_i, v_i) \in P$, where $1 \leqslant i \leqslant n$, we define the following sets of affine functions in dimension three:
  1. $(3^{|u_i|} \cdot x_1 + \sigma(u_i), (n+1) \cdot x_2 + i, 2 \cdot x_3) \in F_1$ for all $1 \leqslant i \leqslant n$,
  2. $(3^{|u_i|} \cdot x_1 + \sigma(u_i), (n+1) \cdot x_2 + i, 2 \cdot x_3 + 1) \in F_2$ for some $1 \leqslant i \leqslant n$, and
  3. $\left(\frac{1}{3^{|v_i|}} \cdot (x_1 - \sigma(v_i)), \frac{1}{n+1} \cdot (x_2 - i), 2 \cdot x_3 - 1\right) \in F_3$ for all $1 \leqslant i \leqslant n$.

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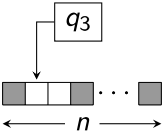
- First construct a word $u' = u_{i_1} u_{i_2} \cdots u_{i_{k-1}}$, where $1 \leqslant i_j \leqslant n$ for all $1 \leqslant j \leqslant k-1$, in the first dimension.

Linear Bounded Automaton

Lemma

The reachability problem for maps over $Aff_{\mathbb{Z}}[\overline{\mathbf{x}}]^n \setminus Add_{\mathbb{Z}}$ *is* PSPACE-*hard.*

- A linear bounded automaton (LBA) is a Turing machine with a finite tape whose length is bounded by a linear function of the size of the input.
- A configuration is $[q, i, w]$, where $q \in Q$, i is the position of the head, $w \in \{0, 1\}^n$ is the word written on the tape.
- The reachability problem: $[q_0, 1, 0^n] \to^* [q_f, 1, 0^n]$?



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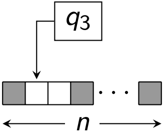

Known fact

The reachability problem for LBAs is PSPACE-complete.

PSPACE-hardness over $\mathrm{Aff}_{\mathbb{Z}}[\overline{\mathbf{x}}]^n \setminus \mathrm{Add}_{\mathbb{Z}}$

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The reachability problem for maps over $\mathit{Aff}_{\mathbb{Z}}[\overline{\mathbf{x}}]^n \setminus \mathit{Add}_{\mathbb{Z}}$ *is* PSPACE*-hard.*

Proof sketch.

- Reduce the reachability problem of an LBA $\mathcal{A}$ to the reachability problem for maps over $\mathrm{Aff}_{\mathbb{Z}}[\overline{\mathbf{x}}]^{k+1} \setminus \mathrm{Add}_{\mathbb{Z}}$.

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- Store the tape content of the LBA $\mathcal{A}$ in the first k dimensions and the current state in the last dimension of the affine map.

PSPACE-hardness over $\mathrm{Aff}_{\mathbb{Z}}[\overline{\mathbf{x}}]^n \setminus \mathrm{Add}_{\mathbb{Z}}$ (continue)

Proof sketch. (continue)

- Let $[q_j, i, w]$ be the current configuration of $\mathcal{A}$.

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Example

The affine function corresponding to $(q_{j_1}, 0, q_{j_2}, 1, L)$ is

$$(x, \ldots, x, 2x+1, x, \ldots, x, a \cdot x + b),$$

where $a \cdot x + b$ corresponds to the edge $((q_{j_1}, i), (q_{j_2}, i-1))$ in $G_{\mathcal{A}}$, and $2x+1$ is in the ith dimension.

Main Results

Theorem

If the dimension n is not fixed, then the reachability problem for maps over $Aff_{\mathbb{Z}}[\overline{\mathbf{x}}]^n \setminus Add_{\mathbb{Z}}$ *is* PSPACE*-complete.*

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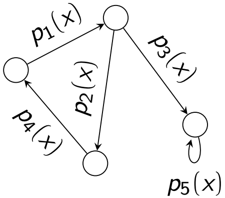
Maps as Language Acceptors

- Let's extend our models to operate on words.



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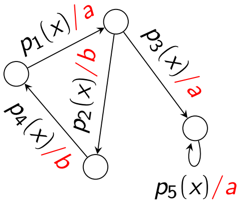

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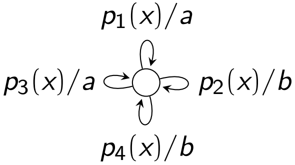

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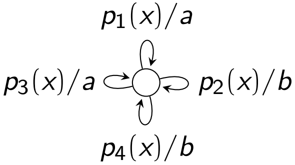

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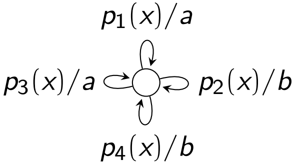

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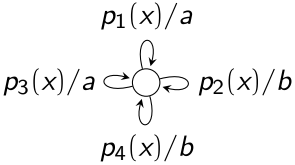

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- In this context, the reachability problems of the previous sections can be seen as language emptiness problem.
- The language accepted by the map is empty if and only if the final configuration is not reachable from the initial configuration.

Universality is Undecidable over $\text{Aff}_{\mathbb{Z}}[\overline{\mathbf{x}}]^2$

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- Let $\mathcal{A}^{\Upsilon}$ be an integer weighted automaton over alphabet Σ for which the universality problem is undecidable [Halava & Harju, 1998].

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- Then, a word $w \in \Sigma^*$ is accepted by the map if and only if the register values $(0, m-1)$ are reachable from $(0, 0)$ while reading word w.

Intersection Emptiness is Undecidable over $\text{Aff}_{\mathbb{Z}}[\overline{\mathbf{x}}]^2$

Definition (Reachability set of a map)

Let $F \subseteq \mathbb{Z}[\overline{\mathbf{x}}]^n$ be a map over $\mathbb{Z}[\overline{\mathbf{x}}]^n$ and let $\overline{\mathbf{x}}_0 \in \mathbb{Z}^n$ be the initial value. The *reachability set* of F is defined iteratively:

$$\begin{aligned} \text{Reach}_0(F) &= \{\overline{\mathbf{x}}_0\}, \\ \text{Reach}_i(F) &= \{f(\overline{\mathbf{x}}) \mid \overline{\mathbf{x}} \in \text{Reach}_{i-1}(F), f \in F\}, \\ \text{Reach}(F) &= \bigcup_{i=0}^{\infty} \text{Reach}_i(F). \end{aligned}$$

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Lemma

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Theorem

Let $F, G \subseteq \Sigma \times \mathrm{Aff}_{\mathbb{Z}}[\overline{\mathbf{x}}]^2$ and $\overline{\mathbf{x}}_{0_F}, \overline{\mathbf{x}}_{0_G}$ and $\overline{\mathbf{x}}_{f_F}, \overline{\mathbf{x}}_{f_G}$ be the respective initial and target values. It is undecidable whether the intersection of the respective languages is empty.

Complexity Landscape

- Complexity of reachability problems in nondeterministic polynomial maps according to the degrees.

degree / dim.	1		2	3	4
	the leading coefficient				
	$a_1 = \pm 1$	$a_1 \in \mathbb{Z}$			
1	NP-c. [2][1]	NP-h. [2]/PSPACE [1][2]			PSPACE-c. [3][3]
2		NP-h. [2]/?	PSPACE-h./?		PSPACE-h. [3]/?
3		undecid.			undecid. [3]

[1] [2] Haase and Halfon. "Integer Vector Addition Systems with States". *RP 2014*.

[2] [1] Finkel, Göller, and Haase. "Reachability in Register Machines with Polynomial Updates". *MFCS 2013*.

[3] [3] Niskanen. "Reachability problem for polynomial iteration is PSPACE-complete". *RP 2017*.

Complexity Landscape from Different View

- Complexity of reachability problems in affine and polynomial maps with respect to inclusion of polynomials of the form $\pm x + b$.

<table>
<tr><th>type / dim.</th><th colspan="2">affine</th><th colspan="2">polynomial</th></tr>
<tr><th></th><th>$a_1 \neq \pm 1$</th><th>$a_1 \in \mathbb{Z}$</th><th>$a_1 \neq \pm 1$</th><th>$a_1 \in \mathbb{Z}$</th></tr>
<tr><td>1</td><td rowspan="4">NP-h./PSPACE</td><td>NP-h. [2]/PSPACE [1]</td><td rowspan="4">NP-h./PSPACE</td><td>PSPACE-c. [3]</td></tr>
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<tr><td>3</td><td rowspan="3">undecid.</td><td rowspan="3">undecid. [3]</td></tr>
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</table>

Complexity Landscape from Different View

- Complexity of reachability problems in affine and polynomial maps with respect to inclusion of polynomials of the form $\pm x + b$.

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	$a_1 \neq \pm 1$	$a_1 \in \mathbb{Z}$	$a_1 \neq \pm 1$	$a_1 \in \mathbb{Z}$
1		NP-h. [2]/PSPACE [1]		PSPACE-c. [3]
2		NP-h. [2]/?		PSPACE-h. [3]/?
3	NP-h./PSPACE		NP-h./PSPACE	
$\vdots$		undecid.		undecid. [3]
n	PSPACE-c.		PSPACE-c.	

Our goal was also to

Investigate the effect of polynomials of the form $\pm x + b$ on the decidability and complexity of the reachability problems!

Concluding Remarks

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THANK YOU